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Boolean Duals of Virtual Groups

ARLAN RAMSAY

Department of Mathematics, University of Colorado, Boulder, Colorado 80302

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The purpose of this paper is to formulate the notion of virtual group and that of a homomorphism between two in such a way that no null sets are involved. That is, we replace the point set by the measure algebra of Borel sets modulo null sets, and the various functions involved must be described by objects associated to these measure algebras, such as σ-homomorphisms. In the process it is necessary to redefine functions satisfying algebraic identities almost everywhere to get new ones satisfying the identities in a stricter sense. Similar refinements on quasiinvariance of measures are also required.

CONTENTS

INTRODUCTION

In [11] G. W. Mackey mentioned the possibility of reformulating the theory of virtual groups or ergodic groupoids in such a way that the exceptional null sets involved in his definitions would not appear. The present paper gives ways of constructing virtual groups and homomorphisms between them from data given in terms of measure algebras (Borel sets modulo null sets), giving thereby a reformulation of that part of the theory.

It is hoped that some of the results showing that the notions can be defined in measure algebra terms will find applications, perhaps even in the special case, often not mentioned, of second countable locally compact groups. However, it may be that the auxiliary results are more interesting and more useful. For example, in Section 1 we show that tensor products do not exist, in the usual sense, in the category of measure algebras, and the obvious attempt at existence is discussed. Section 2 contains results on actions of groupoids on measures. Also, the main result of Section 6 is that the measure class of a measurable groupoid possesses more by way of invariance properties than the definition requires. This added invariance allows us to prove that the set of elements in a measurable groupoid having a given unit for right and left unit (see definitions below) has a natural locally compact group topology.

To say much more about the various sections would require more technicalities than seem appropriate in an introduction, but it does seem worthwhile to comment on the general techniques. They are measure theoretic algebra rather than topological algebra. One technique is to modify a function satisfying an identity a.e. in a loose sense so that the new function satisfies the identity everywhere or at least everywhere on an inessential contraction. This is the technique used in [9] and Sections 3 and 5 of [14]. Another sort of idea is that some sets which have measure zero relative to the obvious measure may be important for algebraic reasons, so it becomes necessary to find a way to recover them from the sets of positive measure. This applies to ranges of homomorphisms in particular, for even in the group case a closed subgroup usually has measure zero.

Now let us recall some definitions and notations from [14]. An analytic (standard) Borel groupoid is an analytic (standard) Borel space G with a Borel subset $G^{(2)} \subseteq G \times G$ whose projection on either factor is all of G, a Borel multiplication m: $G^{(2)} \to G$ taking (x, y) to xy and a Borel mapping $x \to x^{-1}$, such that the operations m and $(\ )^{-1}$ render G a groupoid. This means that: (1) $x(yz)$ is defined iff $(xy)z$ is defined and then they are the same; (2) if (x, y) and $(y, z) \in G^{(2)}$, then $(x, yz) \in G^{(2)}$; and (3) for x in G, $(xx^{-1})x = x = x(x^{-1}x)$ (to write a product ab we assume $(a, b) \in G^{(2)}$). Thus G is an abstract category with inverses. Thinking of the elements of G as mappings and composing so that $f \circ g(x) = f(g(x))$, xx^{-1} behaves like the identity function on the range of x and $x^{-1}x$ behaves like the identity function on the domain of x, so we designate them by $r(x)$ and $d(x)$, respectively. Now $r(G) = d(G) = U = U_G$ is a Borel subset of G whose elements are called units. If S is an analytic space and $E \subseteq S \times S$ is an analytic set which is an equivalence relation we call E an analytic equivalence relation. Taking $E^{(2)} = \{((x, y), (y, z)): (x, y), (y, z) \in E\}$, $(x, y)(y, z) = (x, z)$ and $(x, y)^{-1} = (y, x)$, E becomes an analytic groupoid. We shall call a pair (G, C) a *measurable groupoid* if G is an analytic groupoid and C is a measure class on G which is invariant under $(\ )^{-1}$ (we say C is symmetric) and has a property called right invariance: Let $C = [\lambda]$ where $\lambda(G) < \infty$, define $\tilde{\lambda} = d_*(\lambda)$ decompose λ over $\tilde{\lambda}$ as $\lambda = \int \lambda_u \, d\tilde{\lambda}(u)$ (see Section 1) so for almost all u, λ_u is carried by $d^{-1}(u)$. If $r(x) = u$, then $(\lambda_u x)(A) = \lambda_u(\{y: d(y) = u \text{ and } yx \in A\})$ defines a measure on $d^{-1}(d(x))$. We ask that there be a $\tilde{\lambda}$-conull set $U_0 \subseteq U$ such that $\lambda_{r(x)}x$ and $\lambda_{d(x)}$ have the same null sets whenever $r(x)$ and $d(x)$ are in U_0. The measure class C is called ergodic if every Borel function f such that $f \circ r = f \circ d$ a.e. relative to λ must be constant a.e. relative to $\tilde{\lambda}$. The measurable groupoid (G, C) is a *virtual group* if C is ergodic.

We call attention to the fact that several results in [14] which were stated for virtual groups are valid for all measurable groupoids (in particular we mention Theorem 5.1 and Lemma 5.2), and will be used in this paper.

1. Measure Theoretic Constructions

After some setting of notation and elementary remarks, in this section we recall the basic result about decompositions of measures and prove a lemma about how decompositions behave under mappings. After that we study tensor products (coproducts) of measure algebras, including relative tensor products. One result is that these do not exist in the usual categorical meaning of the term, but a suitable construction is available. We give this construction and a few of its properties.

For information about standard and analytic Borel spaces, we refer the reader to Sections 33, 34, and 35 of [4], Sections 1–6 of [7] and Chapter 1 of [13]. Some measure theoretic facts are stated as we need them in Section 2 of [14]. We assume all measures are σ-finite, usually finite. If measures λ and μ have the same null sets we write $\lambda \sim \mu$, and for given λ we have $[\lambda] = \{\mu: \mu \sim \lambda\}$. We call a set conull if it has a complement of measure zero, and in that case say that the measure is carried by the set or concentrated on the set. The unit point mass ϵ_x is carried by $\{x\}$.

If $(X, \mathcal{Cl})$ is a Borel space and λ is a measure on $\mathcal{Cl}$, the quotient of $\mathcal{Cl}$ by the σ-ideal of λ-null sets will be denoted $M(\lambda)$ and is a complete Boolean algebra. The measure induced on $M(\lambda)$ will still be denoted by λ, and the quotient σ-homomorphism of $\mathcal{A}$ onto $M(\lambda)$ will be denoted by q, q_λ or maybe q_1 for q_{λ_1} if we need to distinguish several such. If $f\colon (X, \mathcal{A}) \to (Y, \mathcal{B})$ is a Borel function from one Borel space to another and λ is a measure on $\mathcal{A}$ the transform of λ by f or direct image of λ under f is denoted by $f_*(\lambda)$: $f_*(\lambda)(B) = \lambda(f^{-1}(B))$ for $B \in \mathcal{B}$. If μ is another measure on $\mathcal{B}$ and $f_*(\lambda) \ll \mu$, then the formula $f^*(q_\mu(B)) = q_\lambda(f^{-1}(B))$ defines a σ-homomorphism $f^*\colon M(\mu) \to M(\lambda)$. Every such σ-homomorphism arises in this way for analytic spaces; f^* is onto iff f is one–one on some conull set and f^* is one–one iff $f_*(\lambda) \sim \mu$. Also $f^* = g^*$ iff $f = g$ a.e.

LEMMA 1.1. *Let X be a set, $\mathcal{A}_0$ a countable algebra of subsets of X, and $\mathcal{A}$ the σ-algebra generated by $\mathcal{A}_0$. If M is the set of finite measures on $(X, \mathcal{A})$ and has the smallest Borel structure relative to which $\mu \mapsto \mu(A)$ is a Borel function for each $A \in \mathcal{A}$, then $\{(\lambda, \mu)\colon \lambda \ll \mu\}$ is a Borel set in $M \times M$ for the product Borel structure, and hence $\{(\lambda, \mu)\colon \lambda \sim \mu\}$ is also Borel.*

Proof. If λ is a finite measure on $\mathcal{A}$, then $\mathcal{A}_0$ is dense in $\mathcal{A}$ relative to the metric induced by λ. Hence if $\lambda, \mu \in M$ and $\epsilon > 0$, $\delta > 0$, and if $\{A \in \mathcal{A}_0\colon \mu(A) < \delta\} \subseteq \{A \in \mathcal{A}_0\colon \lambda(A) < \epsilon\}$, then we can use sequences converging relative to $\lambda + \mu$ to show that

$$\{A \in \mathcal{A}\colon \mu(A) < \delta\} \subseteq \{A \in \mathcal{A}\colon \lambda(A) \leqslant \epsilon\}.$$

It follows that

$$\{(\lambda, \mu)\colon \lambda \ll \mu\} = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{A \in \mathcal{A}_0} \left(\left\{(\lambda, \mu)\colon \lambda(A) \leqslant \frac{1}{k}\right\} \cup \left\{(\lambda, \mu)\colon \mu(A) \geqslant \frac{1}{n}\right\}\right).$$

In Section 2 of [14] we quoted a decomposition theorem for measures in less generality than we needed, and will correct that here. Also we want to point out a simple proof of the basic theorem, in Lemma 4.4 of [2]. First, let (X, λ) be a standard finite measure space, let Y be an analytic quotient space of X, let p be the quotient mapping of X onto Y and suppose $\mu \sim p_*(\lambda)$. Then there is a function $y \mapsto \lambda(p, y) = \lambda_y$ from Y to the Borel measures on X such that:

(a) for $y \in Y$, λ_y is carried by $p^{-1}(y)$;

(b) if f is a positive Borel function on X, then $y \mapsto \int f\, d\lambda_y$ is Borel on Y and $\int f\, d\lambda = \int (\int f\, d\lambda_y)\, d\mu(y)$.

Several comments are now in order. We call such a family of measures a decomposition of λ over μ relative to p. Any two such decompositions agree a.e. relative to μ, but occasionally (Section 6) small changes matter. Ordinarily only one measure μ will be involved in a given context, so we do not take account of it in the notation $\lambda(p, y)$. If in fact $\nu \sim \mu$ and $g = d\mu/d\nu$, then $y \mapsto g(y)\lambda_y$ gives a decomposition over ν relative to p, and by the uniqueness result we know that any other agrees with this one a.e. If $\mu = p_*(\lambda)$ then almost every λ_y is a probability measure. If on the other hand we have only $p_*(\lambda) \ll \mu$, we can still get a decomposition over μ relative to p by using $dp_*(\lambda)/d\mu$ as above. Also, if $\nu \ll \lambda$ and $g = d\nu/d\lambda$ then we can define $\nu_y(E) = \int g\varphi_E \, d\lambda_y$ to get a decomposition of ν. One consequence of this is that if $\nu \ll \lambda$ then $\nu_y \ll \lambda_y$ a.e. for any decompositions, and if $\nu \sim \lambda$ then $\nu_y \sim \lambda_y$ for μ-almost every y. Another fact, used in constructing fibered products of measures, is that $x \mapsto \int f \, d\lambda_{p(x)}$ is Borel if f is Borel and nonnegative.

Now for the extensions of these facts, first suppose g: $X \to Y$ is only a Borel function, not a quotient mapping. Then $g(X)$ is still analytic, and the Borel structure inherited from Y is the quotient structure on $g(X)$ because they are comparable analytic structures. There is a set $Y_0 \subseteq g(X)$ which is Borel in Y and μ-conull in $g(X)$, and we can let $\lambda_y = 0$ for $y \in Y - Y_0$ to convert the decomposition over $g(X)$ into one over Y. Thus we can get a decomposition relative to g over μ whenever g: $X \to Y$ is a Borel function and $g_*(\lambda) \ll \mu$, where X is standard and Y is analytic. Finally let us show that it is enough for X, Y both to be analytic. Then there is a standard $X_1 \subseteq X$ which is λ-conull. Let $\nu(B) = \lambda(B)$ for $B \subseteq X_1$ and Borel and let $h = g \mid X_1$. Then the decomposition of ν gives measures $\nu(h, y)$ on X_1 and if we define $\lambda(g, y)(B) = \nu(h, y)(B \cap X_1)$ we get a decomposition of λ. Now we are ready for the lemma about transforming decompositions as promised earlier. It is just a generalization of Lemma 11.3 of [6].

LEMMA 1.2. *Let X, Y, Z, W be analytic Borel spaces and let λ be a finite Borel measure on X. Let f: $X \to Y$, p: $X \to Z$, g: $Z \to W$ and q: $Y \to W$ be Borel functions such that $q \circ f = g \circ p$ almost everywhere relative to λ. Set $\mu = f_*(\lambda)$, $\lambda' = p_*(\lambda)$, $\mu' = q_*(\mu) = g_*(\lambda')$. Let $\lambda = \int \lambda_z \, d\lambda'(z)$ and $\lambda' = \int \lambda_w' \, d\mu'(w)$ be decompositions of λ and λ' relative to p and g. For $w \in W$, set $\lambda^w = \int \lambda_z \, d\lambda_w'(z)$. Then $\lambda = \int \lambda^w \, d\mu'(w)$ is a decomposition of λ relative to $g \circ p$ and $\mu = \int f_*(\lambda^w) \, d\mu'(w)$ is a decomposition of μ relative to q. If g is one–one then we can take λ_w' to be a unit point mass at $g^{-1}(w)$ for almost all $w \in W$ and then $\lambda^w = \lambda_{g^{-1}(w)}$ for those w's.*

Proof. For almost every w, λ_w' is concentrated on $g^{-1}(w)$. For λ'-almost every $z \in Z$, λ_z is concentrated on $p^{-1}(z)$, so for μ'-almost every w we know that λ_z is concentrated on $p^{-1}(z) \subseteq p^{-1}(g^{-1}(w))$ for λ_w'-almost every z. Hence for almost every w the measure λ^w is concentrated on $(g \circ p)^{-1}(w)$. Also if h is Borel and nonnegative on X then $z \mapsto \int h\, d\lambda_z$ is Borel and nonnegative on Z so $w \mapsto \int h\, d\lambda^w = \iint h\, d\lambda_z\, d\lambda_w'(z)$ is Borel and nonnegative on W, while

$$\iint h\, d\lambda^w\, d\mu'(w) = \iiint h\, d\lambda_z\, d\lambda_w'(z)\, d\mu'(w)$$

$$= \iint h\, d\lambda_z\, d\lambda'(z)$$

$$= \int h\, d\lambda.$$

Now if A is a Borel set in Y,

$$\mu(A) = \lambda(f^{-1}(A)) = \int \lambda^w(f^{-1}(A))\, d\mu'(w)$$

$$= \int f_*(\lambda^w)(A)\, d\mu'(w).$$

Also, for almost every w relative to μ', $q \circ f = g \circ p$ a.e. relative to λ^w, and for λ^w-almost every x, $g \circ p(x) = w$. Hence for μ'-almost every w the measure $f_*(\lambda^w)$ is concentrated on $q^{-1}(w)$. Thus $\mu = \int f_*(\lambda^w)\, d\mu'(w)$ is a decomposition of μ relative to q.

For the last part, note that the function taking w to a unit point mass at $g^{-1}(w)$ for $w \in g(Y)$ and the 0 measure for $w \notin g(Y)$ is measurable and hence agrees a.e. with a Borel function.

By taking $X = Y$, $Z = W$, and f and g to be identity functions, the basic uniqueness property for decompositions gives a corollary of Lemma 1.2. Since it has a simple direct proof, we give that too.

COROLLARY 1.3. *Let (X, λ), (Y, μ) be analytic Borel spaces with finite measures and let g, h: $X \to Y$ be Borel functions with $g = h$ a.e. and $g_*(\lambda)$, $h_*(\lambda) \ll \mu$. Then $\lambda(g, y) = \lambda(h, y)$ for μ-almost all y.*

Proof. If X_0 is conull and Borel and $g = h$ on X_0, then X_0 is conull for almost every $\lambda(g, y)$ and almost every $\lambda(h, y)$. Set $\lambda(g, y) = \lambda(h, y) = 0$ for a null set of y's so that X_0 is always conull. Then the measures $\lambda(g, y)$ and $\lambda(h, y)$ restricted to X_0 decompose λ restricted to X_0 and hence are almost always the same.

The next subject for discussion in this section is tensor products (sums, coproducts). In the category of Boolean algebras their existence follows easily by duality with Cartesian products of Stone spaces [3, 15], and similar techniques work for σ-algebras [5], but for measure algebras there is no such construction. We will first explain why there is no coproduct for measure algebras and then discuss a construction which almost works and which is sufficient for our needs.

Let λ be Lebesgue measure on the Borel sets $\mathscr{B}$, in $[0, 1] = I$ and let $B_1 = B_2 = M(\lambda)$. If A is a σ-algebra with Stone space X and $\mathscr{A}$ is the σ-algebra generated by the open–closed sets in X there is a natural σ-homomorphism q of $\mathscr{A}$ onto A, and if $\alpha: \mathscr{B} \to A$ is any σ-homomorphism then there is an $\mathscr{A}$-measurable function $g: X \to I$ such that $\alpha = q \circ g^{-1}$. Now suppose A is a probability algebra and that δ_1, δ_2 are σ-homomorphisms of B_1, B_2 into A such that if C is any measure algebra and γ_1, γ_2 are σ-homomorphisms of B_1, B_2 into C then there is a unique σ-homomorphism $\gamma: A \to C$ such that $\gamma \circ \delta_1 = \gamma_1$, $\gamma \circ \delta_2 = \gamma_2$, i.e., suppose $(A; \delta_1, \delta_2)$ is a coproduct of B_1, B_2. Then there exist functions $f_1, f_2: X \to I$ such that $q \circ f_i^{-1} = \delta_i$ $(i = 1, 2)$. If $h(x) = (x, x)$ is the diagonal map of X into $X \times X$, and p_1, p_2 are the projections of $I \times I$ onto I, then we can let $f = f_1 \times f_2 \circ h: X \to I \times I$ and notice that $p_i \circ f = f_i$, so f is $\mathscr{A}$-measurable. If $\psi = q \circ f^{-1}$ then $\psi \circ p_i^{-1} = q \circ f_i^{-1} = \delta_i \circ q_\lambda$. Now let us consider some measure algebras C which test the coproduct property. Let ν be a Borel measure on $I \times I$ such that $p_{1*}\nu + p_{2*}\nu \ll \lambda$. Then we can define $\gamma_i: B_i \to M(\nu) = C$ by $\nu_i(q_\lambda(E)) = q_\nu(p_i^{-1}(E))$. Let $\gamma: A \to C$ be the σ-homomorphism whose existence is guaranteed by the coproduct property. Then for each i, $\gamma \circ \psi \circ p_i^{-1} = \gamma \circ \delta_i \circ q_\lambda = \gamma_i \circ q_\lambda = q_\nu \circ p_i^{-1}$. Thus $\gamma \circ \psi$ and q_ν agree on $\{p_1^{-1}(E): E \in \mathscr{B}\} \cup \{p_2^{-1}(E): E \in \mathscr{B}\}$, which generates $\mathscr{B}(I \times I)$ as a σ-algebra, so they are the same. Now $\mu \circ \psi(E) = 0$ implies $\psi(E) = 0$, which implies $q_\nu(E) = \gamma \circ \psi(E) = 0$. Hence $\nu \ll \mu \circ \psi$ whenever ν is a measure on $I \times I$ whose projections are absolutely continuous relative to λ. If J is any line segment in $I \times I$ of slope 1 and length greater than zero, we can take ν to be proportional to the length measure on J. It follows that $\mu \circ \psi$ must give positive measure to any such J, and since there are uncountably many disjoint segments J, we arrive at a contradiction. Hence B_1, B_2 have no coproduct.

The construction we need is dual to that of products of measure spaces. If $(X_1, \lambda_1)(X_2, \lambda_2)$ are standard measure spaces, $B_1 = M(\lambda_1)$, $B_2 = M(\lambda_2)$ and $B = M(\lambda_1 \times \lambda_2)$, then the σ-homomorphisms β_1, β_2 taking $q_1(E_1)$ to $q(E_1 \times X_2)$ and $q_2(E_2)$ to $q(X_1 \times E_2)$ are isomorphisms of B_1, B_2 into B. Then $\beta_1(q_1(E_1)) \wedge \beta_2(q_2(E_2)) = q(E_1 \times E_2)$, so $\beta_1(B_1)$

and $\beta_2(B_2)$ are independent subalgebras of B. Therefore, if B_0 is the subalgebra of B generated by $\beta_1(B_1) \cup \beta_2(B_2)$, then (B_0, β_1, β_2) is an algebraic tensor product of B_1, B_2. We will write $e_1 \otimes e_2$ for the element $\beta_1(e_1) \wedge \beta_2(e_2)$. The triple (B, β_1, β_2) serves as the best measure algebra substitute for a tensor product of B_1, B_2, and since the algebraic tensor product will not occur, the notation $B_1 \otimes B_2$ will be used for B, even though it is not really a tensor product. A more general construction corresponding to relative products, or relative tensor products, will also be necessary and we now proceed to that construction.

Let (X_1, λ_1), (X_2, λ_2) and (Y, μ) be standard measure spaces, set $B_i = M(\lambda_i)$, $A = M(\mu)$ and suppose $\epsilon_1: A \to B_1$, $\epsilon_2: A \to B_2$ are imbeddings. We want a measure algebra $B = M(\lambda)$ and imbeddings $\beta_1: B_1 \to B$, $\beta_2: B_2 \to B$ such that $\beta_1 \circ \epsilon_1 = \beta_2 \circ \epsilon_2$, while $\beta_1(B_1)$ and $\beta_2(B_2)$ are independent over $\beta_1 \circ \epsilon_1(A)$ (the latter is not really relevant). This is essentially dual to a fibered product situation and a construction was given in [14] which goes as follows. Let p_1, p_2 be Borel functions of X_1, X_2 into Y inducing the inclusions ϵ_1, ϵ_2. The measure λ is going to be carried by the set

$$X_1 \times' X_2 = \{(x_1, x_2) \in X_1 \times X_2 : p_1(x_1) = p_2(x_2)\}.$$

Decompose $\lambda_1 = \int \lambda_{1y}\, d\mu(y)$, $\lambda_2 = \int \lambda_{2y}\, d\mu(y)$. If f is a nonnegative Borel function on $X_1 \times X_2$ define $f^0: Y \to \mathbb{R}$, $f^1: X_1 \to \mathbb{R}$, $f^2: X_2 \to \mathbb{R}$ by

$$f^0(y) = \int f\, d\lambda_{1y} \times \lambda_{2y},$$

$$f^1(x_1) = \int f(x_1, x_2)\, d\lambda_{2p_1(x_1)}(x_2),$$

$$f^2(x_2) = \int f(x_1, x_2)\, d\lambda_{1p_2(x_2)}(x_1).$$

Then f^0, f^1, f^2 are Borel functions because they are if $f(x_1, x_2) = g(x_1)\, h(x_2)$ with g and h nonnegative and Borel, and the set of f's for which they are Borel is closed under monotone limits and linear operations. Thus we can define $\lambda = \int \lambda_{1y} \times \lambda_{2y}\, d\mu(y)$ and compute

$$\begin{aligned} \int f^1\, d\lambda_1 &= \iiint f(x_1, x_2)\, d\lambda_{2p_1(x_1)}(x_2)\, d\lambda_{1y}(x_1)\, d\mu(y), \\ &= \int_Y \int_{p_1^{-1}(y)} \int_{p_2^{-1}(y)} f(x_1, x_2)\, d\lambda_{2y}(x_2)\, d\lambda_{1y}(x_1)\, d\mu(y), \\ &= \int_Y f^0\, d\mu = \int f\, d\lambda. \end{aligned}$$

Writing ϵ_x for the point mass at x, this shows $\lambda = \int \epsilon_{x_1} \times \lambda_{2p_1(x_1)}\, d\lambda_1(x_1)$. By symmetry we get $\int f\, d\lambda = \int f^2\, d\lambda_2$, so $\lambda = \int \lambda_{1p_2(x_2)} \times \epsilon_{x_2}\, d\lambda_2(x_2)$. Several facts are fairly easy to check: $\lambda(X_1 \times X_2 - X_1 \times' X_2) = 0$, λ depends only on λ_1, λ_2, μ, ϵ_1, and ϵ_2 and not on the particular decompositions of λ_1, λ_2 or choice of p_1, p_2 so we will denote it by $\lambda_1 \times' \lambda_2$ (ignoring the μ and ϵ_1, ϵ_2 in our notation). The measure class $[\lambda_1 \times' \lambda_2]$ depends only on $[\lambda_1]$, $[\lambda_2]$, ϵ_1, and ϵ_2 and we will write $M(\lambda_1) \otimes'_{\epsilon_1,\epsilon_2} M(\lambda_2)$ for $M(\lambda_1 \times' \lambda_2)$, or more often simply $M(\lambda_1) \otimes' M(\lambda_2)$ for typographical simplicity. If $\lambda_1(E_1) > 0$ then $\lambda(E_1 \times X_2) > 0$ while if $\lambda_2(E_2) > 0$ then $\lambda(X_1 \times E_2) > 0$ so that the functions β_1 taking $q_1(E_1)$ to $q(E_1 \times X_2)$ and β_2 taking $q_2(E_2)$ to $q(X_1 \times E_2)$ are imbeddings of $M(\lambda_1)$ and $M(\lambda_2)$ into $M(\lambda)$. If $e_1 = q_1(E_1)$ and $e_2 = q_2(E_2)$ we may write $e_1 \otimes' e_2$ for $q(E_1 \times E_2)$, and then $\beta_1(e_1) = e_1 \otimes' 1$, $\beta_2(e_2) = 1 \otimes' e_2$. Note that the elements $e_1 \otimes' e_2$ generate $M(\lambda_1) \otimes' M(\lambda_2)$. If E is a Borel set in Y, then $(p_1^{-1}(E) \times X_2) \cap X_1 \times' X_2 = (X_1 \times p_2^{-1}(E)) \cap (X_1 \times' X_2)$, from which it follows that $\beta_1 \circ \epsilon_1 = \beta_2 \circ \epsilon_2$. The independence of $\beta_1(B_1)$ and $\beta_2(B_2)$ over $\beta_1 \circ \epsilon_1(A)$ does not matter; Theorem 1.6 is related to this and will serve our purposes.

Notice that if $\nu_1 \ll \lambda_1$ and $\nu_2 \ll \lambda_2$ then $\nu_{1y} \ll \lambda_{1y}$ and $\nu_{2y} \ll \lambda_{2y}$ for almost all y and hence $\nu_1 \times' \nu_2 \ll \lambda_1 \times' \lambda_2$. The next lemma and its corollary are easy, but important for proving that weaker version of the functorial property of relative tensor products which is true here.

LEMMA 1.4. *Let X_1, X_2, Y_1, Y_2, Z be analytic Borel spaces and let λ, μ, ν be finite Borel on X_1, Y_1, Z respectively. Suppose that $f\colon X_1 \to X_2$, $g\colon Y_1 \to Y_2$, $p_i\colon X_i \to Z$, $q_i\colon Y_i \to Z$ are Borel functions with $p_2 \circ f = p_1$, $q_2 \circ g = q_1$, and that $p_{1*}(\lambda) \sim q_{1*}(\mu) \sim \nu$. Then $(f \times g)_*(\lambda \times' \mu) = f_*(\lambda) \times' g_*(\mu)$.*

Proof. For ordinary products this follows by evaluating the measures on measurable rectangles. If $\lambda = \int \lambda_z\, d\nu(z)$ is a decomposition of λ then by Lemma 1.2 $f_*(\lambda) = \int f_*(\lambda_z)\, d\nu(z)$ is a decomposition of $f_*(\lambda)$, and similarly for μ and $g_*(\mu)$. Hence, using Lemma 1.2 again,

$$\begin{aligned}(f \times g)_*(\lambda \times' \mu) &= (f \times g)_* \left(\int \lambda_z \times \mu_z\, d\nu(z)\right)\\ &= \int f_*(\lambda_z) \times g_*(\mu_z)\, d\nu(z)\\ &= f_*(\lambda) \times' g_*(\mu).\end{aligned}$$

COROLLARY 1.5. *If* λ_2, μ_2 *are measures on* X_2, Y_2 *with* $f_*(\lambda) \ll \lambda_2$, $g_*(\mu) \ll \mu_2$, *then* $(f \times g)_*(\lambda \times' \mu) \ll \lambda_2 \times' \mu_2$. *If* $f_*(\lambda) \sim \lambda_2$ *and* $g_*(\mu) \sim \mu_2$, *then* $(f \times g)_*(\lambda \times' \mu) \sim \lambda_2 \times' \mu_2$.

THEOREM 1.6. *Suppose that for* $i = 1, 2$ *we have analytic finite measure spaces* (X_i, λ_i) *and* (Y_i, μ_i) *and* σ*-homomorphisms* α_i: $M(\mu_i) \to M(\lambda_i)$. *Let* (Z, ν) *be an analytic finite measure space and let* ϵ_i: $M(\nu) \to M(\mu_i)$, δ_i: $M(\nu) \to M(\lambda_i)$ *be imbeddings such that* $\alpha_i \circ \epsilon_i = \delta_i$ *for* $i = 1, 2$. *Then there is a unique* σ*-homomorphism* α: $M(\mu_1) \otimes' M(\mu_2) \to M(\lambda_1) \otimes' M(\lambda_2)$ *such that* $\alpha(e_1 \otimes' e_2) = \alpha_1(e_1) \otimes' \alpha_2(e_2)$ *for* $e_1 \in M(\mu_1)$, $e_2 \in M(\mu_2)$. *If each* α_i *is an isomorphism, so is* α.

Proof. For $i = 1, 2$, let p_i induce δ_i, let p_{i+2} induce ϵ_i and let f_i induce α_i. Then $p_1 = p_3 \circ f_1$ a.e. and $p_2 = p_4 \circ f_2$ a.e. Then Lemma 1.4 and Corollary 1.5 imply that $(f_1 \times f_2)^*$ exists, and it is not hard to show that it will serve for α. The uniqueness of α follows from the fact that the elements $e_1 \otimes' e_2$ generate $M(\mu_1) \otimes' M(\mu_2)$.

If each α_i is one–one then $f_{1*}(\lambda_1) \sim \mu_1$ and $f_{2*}(\lambda_2) \sim \mu_2$ so $(f_1 \times f_2)_*(\lambda_1 \times' \lambda_2) \sim \mu_1 \times' \mu_2$, and hence α is one–one. If each α_i is onto, then each f_i is one–one on a conull set and hence $f_1 \times f_2$ is one–one on a $\lambda_1 \times' \lambda_2$-conull set, so that α is onto.

Remarks. (a) It is clear from this that $\alpha_1(M(\mu_1)) \cup \alpha_2(M(\mu_2))$ generates $\alpha(M(\mu_1) \otimes' M(\mu_2))$ as a σ-algebra.

(b) An essential uniqueness of $M(\mu_1) \otimes' M(\mu_2)$ as a construction follows from Theorem 1.6.

(c) We write $\alpha_1 \otimes' \alpha_2$ for α.

In Section 7 we have occasion to use a test for when a measure is equivalent to a fibered product.

LEMMA 1.7. *Let* X, Y, Z, W *be analytic Borel spaces, and let* f: $X \to Y$, g: $X \to Z$, p: $Y \to W$, q: $Z \to W$ *be Borel functions with* $p \circ f = q \circ g$. *Let* λ *be a finite measure on* X *and set* $\mu = f_*(\lambda)$, $\nu = g_*(\lambda)$, $\pi = p_*(\mu) = q_*(\nu)$. *Then* (f, g): $X \to Y \times Z$ *induces an isomorphism of* $M(\mu \times' \nu)$ *onto* $M(\lambda)$ *iff* $f_*(\lambda(g, z)) \sim \mu(p, q(z))$ *for* μ*-almost every* z *and* (f, g) *is one–one a.e.*

Proof. (f, g) induces an imbedding iff $(f, g)_*(\lambda) \sim \mu \times' \nu$. If p_2 is the projection of $Y \times Z$ onto Z, then according to Lemma 1.2 we can take $(f, g)_*(\lambda)(p_2, z) = (f, g)_*(\lambda(g, z)) = f_*(\lambda_z) \times \epsilon_z$. Also $\mu \times' \nu(p_2, z) = \mu(p, q(z)) \times \epsilon_z$ by construction of $\mu \times' \nu$. Thus the first condition is equivalent to $(f, g)_*(\lambda) \sim \mu \times' \nu$, i.e. $(f, g)^*$ exists and is one–one, while the second condition is equivalent to $(f, g)^*$ being onto.

As a final remark, we point out that fibered products of three or more measures are associative, in the same sense that ordinary products are. If we have λ_1, λ_2, λ_3, μ on X_1, X_2, X_3, Y and $p_i\colon X_i \to Y$ for $i = 1, 2, 3$, with $p_{i*}(\lambda_i) \sim \mu$, then

$$(\lambda_1 \times' \lambda_2)(p_1 \times p_2, y) = \lambda_1(p_1, y) \times \lambda_2(p_2, y)$$

and similarly for $\lambda_2 \times' \lambda_3$, so that $(\lambda_1 \times' \lambda_2) \times' \lambda_3$ and $\lambda_1 \times' (\lambda_2 \times' \lambda_3)$ are both given by $\int \lambda_{1y} \times \lambda_{2y} \times \lambda_{3y}\, d\mu(y)$.

2. Measure Theory of Groupoid Actions

In this section we develop some of the tools for dealing with transformations of measures on spaces on which a groupoid acts. These actions of course need to be Borel actions. The case of a group action provides the motivation, and suggests other results, but we develop only the generalizations needed in the rest of the paper.

Let S be an analytic Borel space and let G be an analytic Borel groupoid, and let G act *on* S on the right [14], so that for each $x \in G$, there are Borel sets $D(x)$, $R(x) \subseteq S$ and that $s \mapsto sx$ is a Borel isomorphism of $D(x)$ onto $R(x)$, and $\bigcup \{D(x)\colon x \in G\} = S$. Also we require that $F = \{(s, x) \in S \times G\colon sx \text{ is defined}\}$ to be Borel in $S \times G$ and $(s, x) \mapsto sx$ to be Borel. In this setting we can consider the action of G on certain measures on S, generalizing the action of a group on measures defined on a set on which the group acts. If ν is carried by $D(x)$ then $\nu \cdot x$ can be defined by $\nu \cdot x(A) = \nu(\{s \in D(x)\colon sx \in A\})$. Actually the formula makes sense anyway, but if $\nu(S) > 0$ and $\nu(D(x)) = 0$, then $\nu \cdot x = 0$ even though $\nu \neq 0$, which is not acceptable in a groupoid action since the transformations would not be one–one.

Lemma 2.1. *Let $\mathscr{M} = M(S)$ denote the set of finite Borel measures on S and for $\nu \in \mathscr{M}$ and $x \in G$ define $\nu \cdot x(A) = \nu(\{s \in D(x)\colon sx \in A\})$ whenever $A \subseteq S$ is Borel. Then $(\nu, x) \mapsto \nu \cdot x$ is a Borel function from $\mathscr{M} \times G$ to $\mathscr{M}$. If $\mathscr{M}_0 = \{\nu \in \mathscr{M}\colon$ for some $x \in G$, $\nu(S - D(x)) = 0\}$, then $\mathscr{M}_0$ is analytic in S and the function $(\nu, x) \mapsto \nu \cdot x$ from $\mathscr{M}_0 \times' G = \{(\nu, x)\colon \nu(S - D(x)) = 0\}$ to $\mathscr{M}_0$ defines a (Borel) action of G on $\mathscr{M}_0$.*

Proof. For $(s, x) \in F = \{(s, x): s \in D(x)\}$ we define $\tau(s, x) = (sx, x^{-1})$, and note that τ is a Borel automorphism of F. We can also regard τ as a Borel function from F to $S \times G$. Next, notice that $(\nu_1, \nu_2) \mapsto \nu_1 \times \nu_2$ is a Borel function from $\mathscr{M}(S) \times \mathscr{M}(G)$ to $\mathscr{M}(S \times G)$ and that $x \to \epsilon_x$ is Borel from G to $\mathscr{M}(G)$ so that $(\nu, x) \mapsto \nu \times \epsilon_x$ is Borel from $\mathscr{M} \times G$ to $\mathscr{M}(S \times G)$. Now if A is Borel in S and $(\nu, x) \in \mathscr{M} \times G$, we have

$$\begin{aligned}(\nu \cdot x)(A) &= \nu(\{s \in S: sx \in A\}) \\ &= \nu \times \epsilon_x(\tau^{-1}(A \times G)),\end{aligned}$$

so that $(\nu, x) \mapsto (\nu \cdot x)(A)$ is a Borel function on $\mathscr{M} \times G$. Hence $(\nu, x) \mapsto \nu \cdot x$ is a Borel function.

Now $\nu(S - D(x)) = 0$ iff $(\nu \cdot x)(S) = (\nu \cdot x)(R(x)) = \nu(D(x)) = \nu(S)$, so $\{(\nu, x): \nu(S - D(x)) = 0\}$ is a Borel subset of $\mathscr{M} \times G$. Since $\mathscr{M}_0$ is the projection into $\mathscr{M}$ of this set, $\mathscr{M}_0$ must be analytic, and $\{(\nu, x): \nu(S - D(x)) = 0\}$ is a Borel set in $\mathscr{M}_0 \times G$.

From the way measures transform under functions in general, it is clear that if $(x, y) \in G^{(2)}$ and $\nu(D(x)) = \nu(S)$ then $(\nu \cdot x)(D(y)) = (\nu \cdot x)(R(x)) = \nu(S)$ and $(\nu \cdot x) \cdot y = \nu \cdot (xy)$ and that if u is a unit and $\nu(D(u)) = \nu(S)$ then $\nu \cdot u = \nu$.

The question arises as to whether $\mathscr{M}_0$ is not in fact a Borel set in $\mathscr{M}$. To answer this in the affirmative we seem to need the action to be one which respects a partition of S, where the partition corresponds to a Borel function from S onto another analytic space. The next lemma is used in the proof.

LEMMA 2.2. *Let T be a countably separated Borel space. Then $P = \{\mu \in \mathscr{M}(T)$: for some t, $\mu(T - \{t\}) = 0\}$ is a Borel set in $\mathscr{M}(T)$.*

Proof. The Borel structure on $\mathscr{M}(T)$ is the smallest relative to which $\mu \to \mu(A)$ is Borel for each Borel set $A \subseteq T$. If $E_1, E_2, \ldots,$ are Borel sets which separate points in T, then for each n the set $M_n = \{\mu: \mu(E_n)\, \mu(T - E_n) = 0\}$ is a Borel set. It is easy to show that $P = \bigcap \{M_n: n = 1, 2, \ldots\}$.

DEFINITION 2.3. Let G be an analytic Borel groupoid acting on an analytic Borel space S in a Borel manner. We call it a partition action if there are an analytic Borel space T and Borel functions p of S onto T and q of G onto T such that $F = \{(s, x); p(s) = q(x)\}$, so that $D(x) = p^{-1}(q(x))$ for $x \in G$. We also say the action is partitioned over T.

LEMMA 2.4. *If the action of G on S is a partition action, then $\mathcal{M}_0$ is a Borel set in $\mathcal{M}(S)$.*

Proof. Let P, q, T be taken from the definition of partition action. If A is a Borel set in T, then $p^{-1}(A)$ is Borel in S so $\nu \mapsto p_*(\nu)(A) = \nu(p^{-1}(A))$ is a Borel function on $\mathcal{M}(S)$. The level sets of p are the sets $D(x)$ for $x \in G$, so $\mathcal{M}_0 = \{\nu \in \mathcal{M}(S): p_*(\nu)$ is concentrated on one point$\}$. Since T is countably separated, $\mathcal{M}_0$ is the inverse image under a Borel function of a Borel set.

COROLLARY 2.5. *If the action of G on S is true, then $\mathcal{M}_0$ is a Borel set.*

Proof. If $s \in S$, $u_1, u_2 \in U$ and su_1, su_2 are defined then $(su_1)u_2$ is defined so u_1u_2 is defined and $u_1 = u_2$. Thus there is a function p of S onto U such that $p(s) = r(x)$ iff sx is defined. Now $\{(s, u) \in S \times U: s \in D(u)\} = S \times U \cap F$ and hence is a Borel set W in $S \times U$. If A is Borel in U, $p^{-1}(A)$ is the projection onto S of $(S \times A) \cap W$ and the projection is one–one on W, so p is a Borel function.

We will need a direct consequence of Lemma 2.1; it is the form of the statement that is different, not the essential content. The next lemma helps to set the context.

LEMMA 2.6. *Let the action of G on S be partitioned over T, with $p: S \to T$, $q: G \to T$ the functions involved. Then there is a unique action of G on T such that p is G-equivariant from S to T.*

Proof. If there is an equivariant action, $p(s)x = p(sx)$ whenever $(s, x) \in S \times' G$, i.e., $T \times' G = \{(p(s), x): (s, x) \in S \times' G\}$ and $p(s)x = p(sx)$. This proves uniqueness, and to show existence we must check that this defines an action. First of all, $T \times' G = \{(t, x) \in T \times G: t = q(x)\}$ is the inverse image of the diagonal in $T \times T$ under the Borel function which takes (t, x) to $(t, q(x))$, so $T \times' G$ is Borel in $T \times G$. Next, the Borel structure on $T \times' G$ is the quotient structure from $S \times' G$, and the function $(s, x) \mapsto p(sx)$ is Borel. Now if $p(s_1) = p(s_2)$ and (s_1, x), $(s_2, x) \in S \times' G$, then $s_1, s_2 \in D(x) = p^{-1}(q(x))$ so $s_1x, s_2x \in R(x) = D(x^{-1}) = p^{-1}(q(x^{-1}))$, i.e., $p(s_1x) = p(s_2x)$. Thus $(s, x) \mapsto p(sx)$ factors into the projection of $S \times' G$ onto $T \times' G$ and the "action" $T \times' G \to T$. Since the composition is Borel, so is the "action." The two defining properties of an action (units act properly and $t(xy) = (tx)y$ if $(t, x) \in T \times' G$, $(x, y) \in G^{(2)}$) are inherited from S.

Note that another way to define the action is by $D(x)x = R(x)$, because the elements of T can be regarded as the sets in the partition of S.

LEMMA 2.7. *Let the action of G on S be partitioned over T relative to p, q and suppose ν is a Borel function from $T \times' U = \{(t, u) \in T \times U: t = q(u)\}$ to $\mathscr{M}(S)$. Then $(t, x) \to \nu(t, r(x))x$ is Borel from $T \times' G$ to $\mathscr{M}(S)$. Whenever $\nu(t, r(x))$ is concentrated on $D(x) = p^{-1}(t)$ then $\nu(t, r(x)) \cdot x$ is concentrated on $R(x) = p^{-1}(tx)$.*

LEMMA 2.8. *Let G, S, T, p, q be as in Lemma 2.7. Suppose $(G, [\lambda])$ is a measurable groupoid with $\lambda(G) = 1$ and suppose $\lambda = \int \lambda_u \, d\tilde{\lambda}(u)$ is a decomposition of λ relative to d with $\lambda_u(G)$ always 1. If ν is a Borel function from $T \times' G$ to $\mathscr{M}_1(G \times' G)$ (probability measures on $G \times' G$), then $(t, u) \mapsto \nu'(t, u) = \int \nu(t, x) \, d\lambda_u(x)$ is a Borel function from $T \times' U$ to $\mathscr{M}_1(S \times' G)$.*

Proof. It suffices to show that if $f: T \times' G \to [0, 1]$ is a Borel function then the function $f': T \times' U \to [0, 1]$ defined by $f'(t, u) = \int f(t, x) \, d\lambda_u(x)$ is a Borel function. If $g: T \to \mathbb{R}$ and $h: G \to \mathbb{R}$ are Borel and $f(t, x) = g(t) h(x)$ for $(t, x) \in T \times' G$ the conclusion is clear. Now use the standard reasoning about linearity and monotone limits.

An important notion in the study of actions of groups is that of quasiinvariant measure. The next theorem states that two possible ways of defining the notion are equivalent. For motivation the reader is encouraged to look at Corollary 2.10, where the special case of a group is considered. Half of Corollary 2.10 is contained in Theorem 4.3 of [14]. It should be remarked that the restriction to partition actions made here is for the purpose of allowing the construction of a measure on F from one on S and one on G using the fibered product of measures. Possibly some variation could be used in general. This result is just a generalization of Proposition 2.5 on page 72 of [1].

THEOREM 2.9. *Let $(G, [\mu])$ be a measurable groupoid with $\mu(G) = 1$ and let G have an action on an analytic Borel space S which is partitioned over the analytic Borel space T using $p: S \to T$ and $q: G \to T$. Let τ be the associated Borel automorphism of $F = S \times' G$. If λ is a probability measure on S such that $p_*(\lambda) \sim q_*(\mu)$, define $\lambda \times' \mu = \int \lambda(p, t) \times \mu(q, t) \, dp_*(\lambda)(t)$, where $\lambda(p, \cdot)$ and $\mu(q, \cdot)$ decompose λ and μ over $p_*(\lambda)$ relative to p, q. Then $\lambda \times' \mu$ is τ-quasiinvariant iff for $p_*(\lambda)$-almost all t in T we have $\lambda(p, t)x \sim \lambda(p, tx)$ for $\mu(q, t)$-almost all x in $q^{-1}(t)$, and this happens iff for μ-almost all x we have $\lambda(p, q(x))x \sim \lambda(p, q(x^{-1}))$.*

Proof. We may as well suppose, for use in a moment, that μ is symmetric. If A is a Borel set in $S \times' G$ and $x \in G$, then the x-section is denoted by A_x, and we calculate

$$\begin{aligned}\tau^{-1}(A)_x &= \{(s, x) \in S \times' G: (sx, x^{-1}) \in A\}_x \\ &= \{s \in S: sx \in A_{x^{-1}}\} \\ &= (A_{x^{-1}})x^{-1}.\end{aligned}$$

Also $\lambda \times' \mu = \int \lambda(p, q(x)) \times \epsilon_x \, d\mu(x)$, so

$$\begin{aligned}\tau_*(\lambda \times' \mu)(A) &= \int \lambda(p, q(x); (A_{x^{-1}})x^{-1}) \, d\mu(x) \\ &= \int (\lambda(p, q(x))x)(A_{x^{-1}}) \, d\mu(x) \\ &= \int (\lambda(p, q(x^{-1}))x^{-1})(A_x) \, d\mu(x).\end{aligned}$$

Thus $\tau_*(\lambda \times' \mu) = \int (\lambda(p, q(x^{-1}))x^{-1}) \times \epsilon_x \, d\mu(x)$, and this is equivalent to $\lambda \times' \mu$ iff $\lambda(p, q(x^{-1}))x^{-1} \sim \lambda(p, q(x))$ for μ-almost every x, by the fact that two measures are equivalent iff almost all constituents in their decompositions are equivalent. Hence $\tau_*(\lambda \times' \mu) \sim \lambda \times' \mu$ iff $\lambda(p, q(x))x \sim \lambda(p, q(x^{-1}))$ for μ-almost every x. The equivalence with the other condition holds because $\mu = \int \mu(q, t) \, dp_*(\lambda)(t)$.

COROLLARY 2.10. *Let G be a locally compact group and let S be an analytic Borel G-space. Let μ be a measure in the class of Haar measure and let λ be a finite Borel measure on S. Define $\tau(s, x) = (sx, x^{-1})$ for $(s, x) \in S \times G$. Then λ is G-quasiinvariant iff $\lambda \times \mu$ is τ-quasiinvariant.*

Proof. Here T consists of just one point and $\lambda \times' \mu = \lambda \times \mu$, so each $\lambda(1, t) = \lambda$ and the preceding theorem says that $\lambda \times \mu$ is quasiinvariant iff $\lambda x \sim \lambda$ for μ-almost all x. The set $\{x \in G: \lambda x \sim \lambda\}$ is a subsemigroup of G, and being conull must be all of G.

DEFINITION 2.11. If $(G, [\mu])$ is a measurable groupoid with a right partition action on an analytic Borel space S, we call a measure λ on S quasiinvariant under the action iff any and hence all of the three equivalent conditions of Theorem 2.9 hold.

THEOREM 2.12. *Let $(G, [\mu])$ be an analytic measurable groupoid and let G have an action on an analytic Borel space S which is partitioned* *over the analytic Borel space T using $p\colon S \to T$ and $q\colon G \to T$. If λ is a finite measure on S quasiinvariant under the action, and $\lambda(p, \cdot)$ is a decomposition of λ relative to p, then there is an* i.c. *G_0 of G such that $\lambda(p, q(x)) \cdot x \sim \lambda(p, q(x^{-1}))$ whenever $x \in G_0$.*

Proof. If F is the set of all such x's in G, then F is conull by Theorem 2.9, and it is straightforward to show that F is closed under multiplication. Hence F contains an i.c.

THEOREM 2.13. *Let $(G, [\mu])$ be a measurable groupoid and let S_1, S_2 be analytic Borel spaces on which G has actions partitioned over the analytic Borel space T relative to p_1, q and p_2, q respectively. Let $f\colon S_1 \to S_2$ be an equivariant Borel function ($p_2 \circ f = p_1$ and $f(sx) = f(s)x$ if $(s, x) \in S_1 \times' G$), let λ_1 be a finite measure on S_1 and set $\lambda_2 = f_*(\lambda_1)$. Decompose λ_1 over λ_2 relative to f: $\lambda_1 = \int \lambda_1(f, s)\, d\lambda_2(s)$. Then λ_1 is quasiinvariant iff λ_2 is quasiinvariant and the set $\{(s, x) \in S_2 \times' G\colon \lambda_1(f, s) \cdot x \sim \lambda_1(f, sx)\}$ is $\lambda_2 \times' \mu$-conull.*

Proof. If $\lambda_2 = \int \lambda_2(p_2, t)\, d\lambda_2'(t)$ is a decomposition of λ_2 relative to p_2, where $\lambda_2' = p_{2*}(\lambda_2) = p_{1*}(\lambda_1)$, then we can take $\lambda_1(p_1, t) = \int \lambda_1(f, s)\, d\lambda_2(p_2, t; s)$ to decompose λ_1 over λ_2' relative to p_1. By Lemma 1.2 if $t = q(x)$ we have $\lambda_1(p_1, t)x = \int \lambda_1(f, s)x\, d(\lambda_2(p_2, t)x)(s)$. If this is equivalent to $\lambda_1(p_1, tx)$, then $\lambda_2(p_2, t)x \sim \lambda_2(p_2, tx)$ and also $\lambda_1(f, s)x \sim \lambda_1(f, sx)$ for $\lambda_2(p_2, t)$-almost all s, and conversely. These facts combine to give the desired result.

Remarks. (1) If we define $g(s, x) = (f(s), x)$ for $(s, x) \in S_1 \times' G$, and let τ_1, τ_2 be the automorphisms of $S_1 \times' G$ and $S_2 \times' G$, then $g \circ \tau_1 = \tau_2 \circ g$ and $g_*(\lambda_1 \times' \mu) = \lambda_2 \times' \mu$. This gives another proof that if λ_1 is quasi-invariant then so is λ_2.

(2) This theorem should generalize to mappings between actions partitioned over different spaces, but it will be more complicated and we will not need the result in this paper.

3. BOOLEAN DUALS OF GROUPOIDS

Let (G, C) be a measurable groupoid and suppose $G^{(2)}$ is the domain of the multiplication on G while d and r are the domain and range mappings. Let us consider how to define cogroupoids. For groups the objects $G^{(2)}$, d, r are trivial and they therefore do not enter into the formulation of cogroups. However for cogroupoids it will be necessary to include them explicitly. The mapping $(\ )^{-1}$

taking x to x^{-1} will also be included explicitly in the formulation. Since $G^{(2)} = \{(x, y): d(x) = r(y)\}$ is the inverse image of the diagonal in $U \times U$ ($U = U_G$) under $d \times r$, there is a natural measure class induced on $G^{(2)}$ by C. The construction was given in Section 1 and goes as follows. Let $\lambda \in C$ be a probability measure, which can be taken to be invariant under $(\ )^{-1}$. Let $\tilde{\lambda} = d_*(\lambda) = r_*(\lambda)$ and $\lambda = \int \lambda_u \, d\tilde{\lambda}(u) = \int \lambda^u \, d\tilde{\lambda}(u)$ where λ_u is carried by $d^{-1}(u)$ and λ^u is carried by $r^{-1}(u)$. Then

$$\lambda^{(2)} = \int \lambda_u \times \lambda^u \, d\tilde{\lambda}(u) = \int \lambda_{r(x)} \times \epsilon_x \, d\lambda(x) = \int \epsilon_x \times \lambda^{d(x)} \, d\lambda(x)$$

is a measure on $G \times G$ for which $\lambda^{(2)}(G \times G - G^{(2)}) = 0$, and may be regarded as a measure on $G^{(2)}$. The measure class of $\lambda^{(2)}$ depends only on C and can be denoted by $C^{(2)}$. If $M = M(\lambda)$, $\tilde{M} = M(\tilde{\lambda})$, $\delta = d^*$ and $\rho = r^*$, then $M^{(2)} = M(\lambda^{(2)})$ is the relative tensor product $M \otimes_{\delta,\rho} M$. Now we need to show that the multiplication $m: G^{(2)} \to G$ induces a σ-imbedding $m^*: M \to M^{(2)}$.

LEMMA 3.1. *For a Borel set $E \subseteq G$, $\lambda(E) = 0$ iff $\lambda^{(2)}(m^{-1}(E)) = 0$.*

Proof. It is convenient to write xA for $\{xy: y \in A \text{ and } (x, y) \in G^{(2)}\}$. Then

$$\lambda^{(2)}(m^{-1}(E)) = \int_G \int_{r^{-1}(d(x))} \varphi_{m^{-1}(E)}(x, y) \, d\lambda^{d(x)}(y) \, d\lambda(x)$$

$$= \int_G \lambda^{d(x)}(x^{-1}E) \, d\lambda(x).$$

For almost all x, $\lambda^{d(x)}(x^{-1}E) = 0$ iff $\lambda^{r(x)}(E) = 0$ because C is invariant, so we have: $\lambda^{(2)}(m^{-1}(E)) = 0$ iff $\lambda^{d(x)}(x^{-1}E) = 0$ for almost all x iff $\lambda^{r(x)}(E) = 0$ for almost all x iff $r^{-1}(\{u: \lambda^u(E) > 0\}$ is λ-null iff $\{u: \lambda^u(E) > 0\}$ is $\tilde{\lambda}$-null iff $\lambda(E) = 0$.

From two of the formulas for $\lambda^{(2)}$ it is clear that the projections p_1, p_2 of $G \times G$ onto G carry $\lambda^{(2)}$ to λ, because almost every $\lambda^{d(x)}$ and almost every $\lambda_{r(x)}$ will be a probability measure. Thus $p_1{}^* = j_1$ and $p_2{}^* = j_2$ exist and they are the natural imbeddings of M as factors in the relative tensor product.

If $T_1(x, y) = (xy, y^{-1})$ and $T_2(x, y) = (x^{-1}, xy)$ for $(x, y) \in G^{(2)}$, then T_1 and T_2 are Borel automorphisms of $G^{(2)}$ with $T_1{}^2 = T_2{}^2 = i$.

LEMMA 3.2. *T_1 and T_2 reserve the measure class of $\lambda^{(2)}$.*

Proof. It will suffice to give the proof for T_1. If E is a Borel set in $G^{(2)}$,

$$T_{1*}\lambda^{(2)}(E) = \lambda^{(2)}(\{(x, y) \in G^{(2)}: (xy, y^{-1}) \in E\})$$

$$= \int_G \lambda_{r(y)}(\{x: (xy, y^{-1}) \in E\}\, d\lambda(y)$$

$$= \int_G \lambda_{r(y)}((E^{y^{-1}})y^{-1})\, d\lambda(y).$$

Now $\lambda^{(2)}(E) = \int_G \lambda_{r(y)}(E^y)\, d\lambda(y)$, and for almost all $y \in G$ we have $\lambda_{r(y)}(A) = 0$ if $\lambda_{d(y)}(Ay) = 0$ whenever $A \subseteq G$ is Borel. It follows that $\lambda^{(2)}(E) = 0$ iff $T_{1*}\lambda^{(2)}(E) = 0$.

These results will show that the proper automorphisms and imbeddings used in defining groupoids make sense. The associative law is more complicated, so it seems worthwhile to take the time to make sure it is properly formulated. The domain of the operation corresponds to $M^{(2)}$, and on the set level we have $G^{(2)} = \{(x, y): d(x) = r(y)\}$. Because $d(xy) = d(y)$ and $r(yz) = r(y)$ whenever xy and yz are defined, the sets $G_1^{(3)} = \{(x, y, z): xy$ is defined and $(xy)z$ is defined$\}$ and $G_2^{(3)} = \{(x, y, z): yz$ is defined and $x(yz)$ is defined$\}$ are both the same as $G^{(3)} = \{(x, y, z): xy$ and yz are defined$\}$: each of them is $\{(x, y, z): d(x) = r(y),\ d(y) = r(z)\}$. By construction, we have $M^{(2)} = M \otimes_{\delta,\rho} M$ with j_1, j_2 the inclusions of M into $M^{(2)}$. Thus we have $j_1 \circ \rho$, $j_2 \circ \delta$, $j_1 \circ \delta$, and $j_2 \circ \rho$ imbedding $\tilde{M}$ into $M^{(2)}$ and also $j_1 \circ \delta = j_2 \circ \rho$. The set $G_1^{(3)}$ corresponds to $M^{(2)} \otimes_{j_1\circ\delta,\rho} M$, the set $G_2^{(3)}$ corresponds to $M \otimes_{\delta,j_1\circ\rho} M^{(2)}$ and $G^{(3)}$ corresponds to a triple relative tensor product $M^{(3)}$ of M over $\tilde{M}$: the inclusions i_1, i_2, i_3 have the properties $i_1 \circ \delta = i_2 \circ \rho$, $i_2 \circ \delta = i_3 \circ \rho$. $M^{(3)}$ also corresponds in a fairly natural way to $M^{(2)} \otimes_{j_1,j_2} M^{(2)}$, a relative tensor product over M. In formulating associativity for cogroupoids we must include the conditions dual to $d(xy) = d(y)$ and $r(xy) = r(x)$. At this point we show only that the various tensor products involved will be isomorphic in a natural way.

To construct the measure $\lambda_1^{(3)} = \lambda^{(2)} \times' \lambda$ whose measure algebra is $M^{(2)} \otimes_{j_2\circ\delta,\rho} M$ we first decompose $\lambda^{(2)}$ relative to $d \circ p_2$. Now $\nu_u = \int \lambda_{r(y)} \times \epsilon_y\, d\lambda_u(y)$ is carried by $(d \circ p_2)^{-1}(u)$ and

$$\int \nu_u\, d\tilde{\lambda}(u) = \int \lambda_{r(y)} \times \epsilon_y\, d\lambda(y) = \lambda^{(2)},$$

so we may take $\lambda_u^{(2)} = \nu_u$. Then $\lambda_1^{(3)} = \int \lambda_u^{(2)} \times \lambda^u\, d\tilde{\lambda}(u)$. Similarly, taking $\lambda^{(2)u} = \int \epsilon_x \times \lambda^{d(x)}\, d\lambda^u(x)$ we get a decomposition of $\lambda^{(2)}$ relative to $r \circ p_1$, so

$$\begin{aligned}\lambda_2^{(3)} &= \int \lambda_u \times \lambda^{(2)u}\, d\tilde{\lambda}(u)\\ &= \iint \lambda_u \times \epsilon_x \times \lambda^{d(x)}\, d\lambda^u(x)\, d\tilde{\lambda}(u)\\ &= \iint \lambda_{r(x)} \times \epsilon_x \times \lambda^{d(x)}\, d\lambda^u(x)\, d\tilde{\lambda}(u)\\ &= \int \lambda_{r(x)} \times \epsilon_x \times \lambda^{d(x)}\, d\lambda(x).\end{aligned}$$

By symmetry, this is also $\lambda_1^{(3)}$, and it is clear that the measure is carried by $\{(x, y, z) \in G \times G \times G\colon d(x) = r(y),\ d(z) = r(y)\}$. The fact that $\lambda_1^{(3)} = \lambda_2^{(3)}$ gives a natural isomorphism of $M^{(2)} \otimes_{j_2 \circ \delta, \rho} M$ with $M \otimes_{\delta, j_1 \circ \rho} M^{(2)}$; let us write $M^{(3)}$ for $M(\lambda_1^{(3)})$.

LEMMA 3.3. *If $\mu \circ \delta = j_2 \circ \delta$ and $\mu \circ \rho = j_1 \circ \rho$, then there exists a σ-imbedding μ^+ from $M^{(2)}$ to $M^{(3)} = M^{(2)}_{j_2 \circ \delta, \rho} \otimes M$ such that $\mu^+(e \otimes 1) = \mu(e) \otimes 1$ for $e \in M$ and a σ-imbedding μ_+ from $M^{(2)}$ to $M^{(3)} = M \otimes_{\delta, j_1 \circ \rho} (u^{(2)})$ such that $\mu_+(1 \otimes e) = 1 \otimes \mu(e)$ for $e \in M$.*

Proof. This follows immediately from Theorem 1.6.

Remark. The hypotheses are the duals of the properties $d(xy) = d(y)$, $r(xy) = r(x)$. This Lemma will make it possible to state the Boolean dual of the associativity condition.

The ergodicity condition is easy to state because of Lemma 4.1 of [14], so we are now ready to give the definition of Boolean cogroupoid and virtual cogroups.

DEFINITION 3.4. A Boolean cogroupoid is a 5-tuple $(M, \tilde{M}, \delta, (\ )^{-1}, \mu)$ where M, $\tilde{M}$ are measure algebras, δ is a σ-imbedding of $\tilde{M}$ into M, $(\ )^{-1}$ is an automorphism of M with period two and μ is a σ-imbedding of M into $M^{(2)} = M \otimes_{\delta, \rho} M$ where $\rho = (\ )^{-1} \circ \delta$. These are subject to several conditions: Let j_1, j_2 be the imbeddings of M into $M^{(2)}$, and set $M^{(3)} = M^{(2)} \otimes_{j_2 \circ \delta, \rho} M$ $(= M \otimes_{\delta, j_1 \circ \rho} M^{(2)})$, then:

(a) $\mu \circ \delta = j_2 \circ \delta$, $\mu \circ \rho = j_1 \circ \rho$ and if μ_+, μ^+ are the σ-homomorphisms $M^{(2)} \to M^{(3)}$ which exist according to Lemma 3.3, then $\mu^+ \circ \mu = \mu_+ \circ \mu$;

(b) there exist automorphisms τ_1, τ_2 of $M^{(2)}$, each with period two, such that

(i) $\tau_1 \circ j_1 = \mu$, $\tau_2 \circ j_2 = \mu$

(ii) $\tau_1 \circ j_2 = j_2 \circ (\ )^{-1}$, $\tau_2 \circ j_1 = j_1 \circ (\ )^{-1}$.

The cogroupoid is ergodic or a *virtual cogroup* iff $e_1, e_2 \in \tilde{M}$, $0 < e_1$ and $\rho(e_2) \geqslant \delta(e_1)$ imply $e_2 = 1$.

THEOREM 3.5. *If $(G, [\lambda])$ is a measurable groupoid, and m: $G^{(2)} \to G$ is the multiplication, then $(M(\lambda), M(\tilde{\lambda}), d^*, (\ )^{-1}, m^*)$ is a Boolean cogroupoid.*

We call this the *cogroupoid dual to G*, or the *Boolean dual of G*. Some properties of groupoids can be seen to correspond to properties of their Boolean duals, by virtue of theorems proved in [14].

THEOREM 3.6. *Let $(G, [\lambda])$ be a measurable groupoid with Boolean dual $B = (M, \tilde{M}, \delta, (\ )^{-1}, \mu)$.*

(a) *$(G, [\lambda])$ is ergodic iff B is ergodic.*

(b) *$(G, [\lambda])$ is essentially transitive iff there is a σ-homomorphism β: $M(\tilde{\lambda}) \otimes M(\tilde{\lambda}) \to M(\lambda)$ such that $\beta \circ j_1 = (\ )^{-1} \circ \delta$ and $\beta \circ j_2 = \delta$.*

(c) *$(G, [\lambda])$ is essentially principal iff $\delta(\tilde{M}) \cup (\ )^{-1}(\delta(\tilde{M}))$ generates M.*

THEOREM 3.7. *If $B = (M, \tilde{M}, \delta, (\ )^{-1}, \mu)$ is a Boolean cogroupoid, then there is a measurable groupoid, whose Boolean dual is isomorphic to B.*

Proof. (This is rather long, so we break it into several steps.)

Step I. Construct an "Almost Groupoid"

Let λ be a probability measure on $I = [0, 1]$ such that $M = M(\lambda)$ with $\lambda(e^{-1}) = \lambda(e)$ for $e \in M$. Let U be a standard Borel space with measure $\tilde{\lambda}$ such that $M(\tilde{\lambda}) = \tilde{M}$ and let d be a Borel function from I onto U such that $\delta = d^*$ and $\tilde{\lambda} = d_*(\lambda)$. There must exist a Borel function γ: $I \to I$ such that $\gamma \circ \gamma =$ identity and $q(E)^{-1} = q(\gamma^{-1}(E))$ for $E \in Bor(I)$. Then we can set $r = d \circ \gamma$ and have $r^* = \rho$, with $\tilde{\lambda} = r_*(\lambda)$. If $\lambda = \int \lambda_u \, d\tilde{\lambda}(u)$ and $\lambda = \int \lambda^u \, d\tilde{\lambda}(u)$ are the decompositions of λ relative to d and r respectively, then we may suppose $\lambda^u = \gamma_*(\lambda_u)$. Then we construct $\lambda^{(2)} = \int \lambda_u \times \lambda^u \, d\tilde{\lambda}(u)$ on I^2 and $M(\lambda^{(2)})$ can be used for $M^{(2)} = M \otimes_{\delta,\rho} M$. Taking $I^{(2)} = \{(x, y) \in I^2 \colon d(x) = r(y)\}$, the σ-imbedding μ: $M \to M^{(2)}$ is induced by a function m: $I^{(2)} \to I$.

Similarly we construct $\lambda^{(3)}$ on I^3, concentrated on $I^{(3)} = \{(x, y, z): r(y) = d(x), d(y) = r(z)\}$ and there are Borel functions m_+, m^+ inducing μ_+, μ^+. From Definition 2.4(a), $m \circ m_+ = m \circ m^+$ a.e. and $m_+(x, y, z) = (m(x, y), z)$ while $m^+(x, y, z) = (x, m(y, z))$ for $\lambda^{(3)}$-almost all (x, y, z). If T_1 and T_2 induce τ_1, τ_2, then from Definition 2.4(b), we can show that $T_1(x, y) = (m(x, y), \gamma(y))$ almost everywhere relative to $\lambda^{(2)}$ and $T_2(x, y) = (\gamma(x), m(x, y))$ a.e. We may also suppose that T_1 and T_2 are chosen so that $T_1{}^2 = T_2{}^2 = i$, so we see that for $\lambda^{(2)}$-almost all $(x, y) \in I^{(2)}$, $m(m(x, y), \gamma(y)) = x$ and $m(\gamma(x), m(x, y)) = y$. Also, $d \circ m(x, y) = d(y)$ and $r \circ m(x, y) = r(x)$ a.e.

From this, we see that there is a $\lambda^{(2)}$-conull Borel set $L \subseteq I^{(2)}$ such that if $(x, y) \in L$, then $m(m(x, y), \gamma(y)) = x$, $m(\gamma(x), m(x, y)) = y$, $T_1(x, y) = (m(x, y), \gamma(y))$, $T_2(x, y) = (\gamma(x), m(x, y))$ and $d(m(x, y)) = d(y)$, while if $(y, z) \in L$, then $m(x, m(y, z)) = m(m(x, y), z)$ for $\lambda_{r(y)}$-almost all x. (By replacing L with $L \cap T_1(L)$ we may arrange that $T_i(L) = L$). This set L will be used extensively in the rest of the proof.

The functions m, d, r, γ almost make I into a groupoid: The appropriate identities hold almost everywhere for the relevant measures, and this suffices for the purpose of constructing a measure theoretic isomorphism with a measurable groupoid.

Step II. "Invariance" of $[\lambda]$

We need to know that $[\lambda]$ is invariant on I in a suitable sense. We have $\lambda = \gamma_*(\lambda)$ by assumption on λ so the only question is about right invariance.

Now $T_1: G^{(2)} \to G^{(2)}$ is Borel and if $p(x, y) = y$ for $(x, y) \in G^{(2)}$, then $p_*(\lambda^{(2)}) = \lambda$ and $\lambda^{(2)} = \int \lambda_{r(y)} \times \epsilon_y \, d\lambda(y)$ is a decomposition of $\lambda^{(2)}$ over λ relative to p. Also $p \circ T_1 = \gamma \circ p$ a.e. relative to $\lambda^{(2)}$ and $\gamma_*(\lambda) = \lambda$. Hence, by Lemma 1.2, $T_{1*}(\lambda_{r(\gamma(y))} \times \epsilon_{\gamma(y)}) \sim \lambda_{r(y)} \times \epsilon_y$ for λ-almost all y. Therefore $T_{1*}(\lambda_{r(y)} \times \epsilon_y) \sim \lambda_{d(y)} \times \epsilon_{\gamma(y)}$ for λ-almost all y. Since $T_{1*}(\lambda_{r(y)} \times \epsilon_y) = m(\cdot, y)_*(\lambda_{r(y)}) \times \epsilon_{\gamma(y)}$ for λ-almost all y, we have $m(\cdot, y)_*(\lambda_{r(y)}) \sim \lambda_{d(y)}$ for λ-almost all y which is the desired right invariance.

Step III. A Mapping ψ of K into a Unitary Groupoid

If $g = dT_{1*}(\lambda^{(2)})/d\lambda^{(2)}$, then for λ-almost all y we have $g = dT_{1*}(\lambda_{d(y)} \times \epsilon_{\gamma(y)})/d(\lambda_{r(y)} \times \epsilon_y)$ and hence

$$m(\ , \gamma(y))_*(\lambda_{d(y)}) \sim \lambda_{r(y)}$$

and

$$g(\ , y) = dm(\ , \gamma(y))_*(\lambda_{d(y)})/d\lambda_{r(y)} \tag{3.1}$$

Choose a conull Borel set $K \subseteq I$ with these properties:

$$y \in K \text{ implies } L^y = \{x\colon (x, y) \in L\} \text{ is } \lambda_{r(y)}\text{-conull and (3.1) holds,} \tag{3.2}$$

and

$$\gamma \circ \gamma = i \text{ on } K \quad \text{ and } \quad \gamma(K) = K. \tag{3.3}$$

Then $y \in K$ implies $L^{\gamma(y)}$ is $\lambda_{d(y)}$-conull, and $x \to m(x, y)$ is one–one from L^y into $d^{-1}(d(y))$. We also will need to know that K can be chosen so that

$$y \in K \text{ implies } \lambda_{d(y)}(I - K) = \lambda_{r(y)}(I - K) = 0. \tag{3.4}$$

To show that this is possible, let K_0 satisfy (3.2) and (3.3) and define $K_1, K_2, \ldots,$ inductively by

$$K_{n+1} = \{y \in K_n\colon \lambda_{d(y)}(I - K_n) = \lambda_{r(y)}(I - K_n) = 0\}.$$

Then each K_n is Borel since $y \to \lambda_{d(y)}$ and $y \to \lambda_{r(y)}$ are, and conull (by induction). Now $d \circ \gamma = r$ on K_0 and $r \circ \gamma = d$ on K_0, so if $\lambda_{d(y)}(I - K_n) = \lambda_{r(y)}(I - K_n) = 0$, then

$$\lambda_{d(\gamma(y))}(I - K_n) = \lambda_{r(\gamma(y))}(I - K_n) = 0.$$

In other words, $\gamma(K_{n+1}) = K_{n+1}$. Set $K = \bigcap_{n=0}^{\infty} K_n$. Then K is Borel and conull, $\gamma(K) = K$, $\gamma \circ \gamma = i$ on K because $K \subseteq K_0$, and (3.2) holds because $K \subseteq K_0$. If $y \in K$, then for every n we have $y \in K_{n+1}$ so $\lambda_{d(y)}(I - K_n) = \lambda_{r(y)}(I - K_n) = 0$. Hence (3.4) holds for K. It follows that

$$\text{if } y \in K, \text{ then } K \cap L^y \text{ is } \lambda_{r(y)}\text{-conull.} \tag{3.5}$$

If $y \in K$, then the inverse of $m(\ , y) \mid L^y$ is $m(\ , \gamma(y))$ by choice of L; also $T_1(x, y) = (m(x, y), \gamma(y))$ for $x \in L^y$, and since $T_1(L) = L$ we have $L^{\gamma(y)} = T_1(L)^{\gamma(y)} = \{m(x, y)\colon x \in L^y\}$. Thus $m(\ , y)$ takes L^y one–one onto $L^{\gamma(y)}$. Since (3.1) holds, a simple calculation shows that the operator $\psi(y)\colon L^2(I, \lambda_{d(y)}) \to L^2(I, \lambda_{r(y)})$ defined by $(\psi(y)f)(x) = g(x, y)^{1/2} f(m(x, y))$ is unitary. If $H_u = L^2(I, \lambda_u)$ for $u \in U$, then $u \to H_u$, $U * H$ is a Hilbert bundle over U, and ψ maps K into the unitary groupoid of the bundle, which we will denote by $\mathscr{U}$ [14]. If i_u is the identity on H_u, $d(\psi(y)) = i_{d(y)}$ and $r(\psi(y)) = i_{r(y)}$.

Let $(|)_u$ denote the inner product in H_u for $u \in U$. If f_1, f_2 are bounded Borel functions on I, regarded as elements of $H_{d(y)}$ and $H_{r(y)}$ respectively, then

$$y \to (\psi(y) f_1 \mid f_2)_{r(y)} = \int f_1(m(x, y))(f_2(x))^- g(x, y)^{1/2} \, d\lambda_{r(y)}(x)$$

$$= \int f_1(m(x, z))(f_2(x))^- g(x, z)^{1/2} \, d(\lambda_{r(y)} \times \epsilon_y)(x, z)$$

is a Borel function of y since $y \to \lambda_{r(y)} \times \epsilon_y$ is Borel. Hence ψ is Borel from K to $\mathscr{U}$. Extend ψ to be constant on $I - K$, taking as value the identity on some H_u.

Next we show that ψ is one–one on K. If $y_1, y_2 \in K$ and $\psi(y_1) = \psi(y_2)$, then $d(y_1) = d(y_2)$ and $r(y_1) = r(y_2)$, while $m(x, y_1) = m(x, y_2)$ for almost all x in $L^{y_1} \cap L^{y_2}$. Hence for some x we have $y_1 = m(\gamma(x), m(x, y_1)) = m(\gamma(x), m(x, y_2)) = y_2$.

Step IV. Construct a Subgroupoid of $\mathscr{U}$ Which is Boolean Isomorphic to the Given Cogroupoid

First set $\nu = \psi_*(\lambda)$ and let G be the subgroupoid of $\mathscr{U}$ generated by $\psi(K)$. We shall show that $(G, [\nu])$ is a measurable groupoid whose dual Boolean cogroupoid is isomorphic to B. Since $\mathscr{U}$ is a standard Borel groupoid the sets G_n defined inductively by $G_1 = \psi(K)$, $G_{n+1} = \{V_1 V_2 \colon (V_1, V_2) \in G_n \times {}'G_n\} \cup \{V^{-1} \colon V \in G_n\}$ are analytic and hence so is $G = \bigcup \{G_n \colon n = 1, 2, \ldots\}$.

For $y \in K$, $\psi(\gamma(y)) \psi(y)$ is a unitary operator on $H_{d(y)}$, which space we can identify with $L^2(L^{\gamma(y)}, \lambda_{d(y)})$, and the operator is obtained by composing with $m(\ , y) \circ m(\ , \gamma(y))$ and multiplying by a positive function. Since the function composed with is the identity on $L^{\gamma(y)}$, the unitary operator is the identity $i_{d(y)}$. Similarly $\psi(y) \psi(\gamma(y))$ is the identity on $H_{r(y)}$, so $\psi(\gamma(y)) = \psi(y)^{-1}$ for $y \in K$. Since $\nu(\mathscr{U} - \psi(K)) = 0$, it follows that $\nu(E^{-1}) = \nu(E)$ for $E \subseteq G$.

If $(y_1, y_2) \in L \cap m^{-1}(K)$, then $\psi(y_1) \psi(y_2)$ is obtained by composing with $m(\ , y_2) \circ m(\ , y_1)$ and then multiplying by a positive function. Since $m(\ , y_2) \circ m(\ , y_1) = m(\ , m(y_1, y_2))$ a.e., we must have $\psi(y_1) \psi(y_2) = \psi(m(y_1, y_2))$. If $y \in K$, $m(x, y) \in K \cap L^{\gamma(y)}$ for $\lambda_{r(y)}$-almost all x in L^y, since $K \cap L^{\gamma(y)}$ is $\lambda_{d(y)}$-conull. Thus if $y \in K$, $\psi(x) \psi(y) = \psi(m(x, y))$ for $\lambda_{r(y)}$-almost all x.

Next notice that for $\tilde{\lambda}$-almost every $u \in U$, λ_u is concentrated on $d^{-1}(u) \cap K$ because of (3.4). If we let $\lambda_u'(E) = \lambda_u(K \cap E)$ then $\lambda = \int \lambda_u' \, d\tilde{\lambda}(u)$ is a decomposition of λ, while $\lambda'_{r(y)} = \lambda_{r(y)}$ and $\lambda'_{d(y)} = \lambda_{d(y)}$ for $y \in K$. Also $\psi_*(\lambda_u')$ is concentrated on $\psi(d^{-1}(u) \cap K)$ and therefore on $d^{-1}(i_u) \cap G$ where i_u is the identity on H_u. Now

$\nu = \int \psi_*(\lambda_u)\, d\tilde{\lambda}(u)$ and $u \to i_u$ carries $\tilde{\lambda}$ to $d_*(\nu)$, so we can take $\nu_{i_u} = \psi_*(\lambda_u')$ in the decomposition of ν relative to d. The decomposition of ν relative to r follows by symmetry with $\lambda^{u'} = (\lambda_u')^{-1}$, and we also get

$$\nu^{(2)} = \int (\psi \times \psi)_*(\lambda_u' \times \lambda^{u'})\, d\tilde{\lambda}(u)$$

$$= \int \psi_*(\lambda_u') \times \psi_*(\lambda^{u'})\, d\tilde{\nu}(i_u)$$

since

$$(\psi \times \psi)_*(\lambda_u' \times \lambda^{u'}) = \psi_*(\lambda_u') \times \psi_*(\lambda^{u'}).$$

This fact, combined with the result of the preceding paragraph shows that the function m_1: $G^{(2)} \to G$ taking (V_1, V_2) to V_1V_2 induces a map $M(\nu) \to M(\nu^{(2)})$ which corresponds to the given map μ: $M(\lambda) \to M(\lambda^{(2)})$, i.e., $\mu \circ \psi^* = (\psi \times \psi)^* \circ m_1^*$.

To show that $[\nu]$ is invariant, we calculate for $y \in K$, and $E \subseteq G$:

$$\begin{aligned}\psi_*(\lambda_{r(y)})(\{V\colon V\psi(y) \text{ is defined and in } E\})\\ &= \lambda_{r(y)}(\{x \in L^y\colon \psi(x)\,\psi(y) \in E\})\\ &= \lambda_{r(y)}(\{x \in L^y\colon m(x, y) \in K \text{ and } \psi(m(x, y)) \in E\})\\ &= \psi_*(m(\ , y)_*(\lambda_{r(y)}))(E).\end{aligned}$$

Since $m(\ , y)_*(\lambda_{r(y)}) \sim \lambda_{d(y)}$ for $y \in K$, $\lambda'_{r(y)} = \lambda_{r(y)}$ and $\lambda'_{d(y)} = \lambda_{d(y)}$, we see that $y \in K$ implies $\nu_{r(\psi(y))} \cdot \psi(y) \sim \nu_{d(\psi(y))}$. Set $F = \{V \in G\colon \nu_{r(V)} \cdot V \sim \nu_{d(V)}\}$. Since $r(V^{-1}) = d(V)$, $V \in F$ implies $V^{-1} \in F$. If $(V_1, V_2) \in (F \times F) \cap G^{(2)}$, then $r(V_2) = d(V_1)$, $r(V_1V_2) = r(V_1)$ and $d(V_1V_2) = d(V_2)$ while

$$\begin{aligned}\nu_{r(V_1)} \cdot V_1V_2 &= (\nu_{r(V_1)} \cdot V_1) \cdot V_2\\ &\sim \nu_{d(V_1)} \cdot V_2\\ &\sim \nu_{d(V_2)}\,.\end{aligned}$$

Thus F is also a subgroupoid. Since $\psi(K) \subseteq F$, we have $G = F$.

Now $(G, [\nu])$ is an analytic Borel groupoid with invariant measure class and ψ^* is an isomorphism of $M(\nu)$ onto M, while $u \to i_u$ induces an isomorphism of $M(\tilde{\nu})$ onto $\tilde{M}$. The various functional identities or almost everywhere identities needed to prove that these maps establish a cogroupoid isomorphism were established above.

THEOREM 3.8. *If (G_1, C_1) and (G_2, C_2) are analytic Borel groupoids with invariant measure classes, and their Boolean duals B_1, B_2 are isomorphic, then G_1 and G_2 have isomorphic inessential contractions.*

Proof. From the isomorphism of B_1, B_2 we deduce that there exist Borel functions $\psi_1: G_1 \to G_2$ and $\psi_2: G_2 \to G_1$ such that $\psi_i(xy) = \psi_i(x)\psi_i(y)$ for almost all $(x, y) \in G_i^{(2)}$ and $\psi_1 \circ \psi_2$ and $\psi_2 \circ \psi_1$ agree a.e. with the appropriate identities. The proof of Theorem 5.1 of [14] applies without ergodicity of the groupoid G, so we may suppose ψ_1 and ψ_2 are (algebraically) homomorphisms on i.c.'s $G_{1,1}$ and $G_{2,1}$. By choosing $G_{2,1}$ first and then $G_{1,1}$ so that $G_{1,1} \subseteq \psi_1^{-1}(G_{2,1})$, we can have $\psi_1(G_{1,1}) \subseteq G_{2,1}$, and then $\psi_2 \circ \psi_1$ is a homomorphism on $G_{1,1}$. The set where $\psi_2 \circ \psi_1$ agrees with the identity is a subgroupoid and being conull it must contain an i.c. $G_{1,2} \subseteq G_{1,1}$. Now $\psi_1(G_{1,2})$ is a conull subgroupoid of $G_{2,1}$ and so contains an i.c. $G_{2,3}$. Set $G_{1,3} = \psi_1^{-1}(G_{2,3})$, and notice that ψ_1 is an isomorphism of $G_{1,3}$ onto $G_{2,3}$, preserving null sets. Finally, if $d(x)$ and $r(x) \in G_{1,3}$ for $x \in G_1$, then $d(\psi(x)) = \psi(d(x))$ and $r(\psi(x)) = \psi(r(x))$ are in $G_{2,3}$ so $\psi(x) \in G_{2,3}$ and hence $x \in G_{1,3}$. Thus $G_{1,3}$ is an i.c.

4. BOOLEAN DUALS OF HOMOMORPHISMS (I)

In this section we begin to study the connection between homomorphisms of measurable groupoids and maps relating their Boolean duals. We consider $\varphi: (G_1, C_1) \to (G_2, C_2)$ and ask for conditions which make the connection an easy one to establish. If $C_1 = [\lambda_1]$ and $C_2 = [\lambda_2]$ one easy situation to study is that in which $\varphi_*(\lambda_1) \ll \lambda_2$ so that $\varphi^*: M(\lambda_2) \to M(\lambda_2)$ exists. We shall say $\varphi_*(C_1) \leqslant C_2$ when $\varphi_*(\lambda_1) \ll \lambda_2$, and write $M([\lambda])$ for $M(\lambda)$ for any measure λ.

THEOREM 4.1. *Let (G_1, C_1) and (G_2, C_2) be measurable groupoids with Boolean duals $B_1 = (M_1, \tilde{M}_1, \mu_1, (C)^{-1}, \delta_1)$ and $B_2 = (M_2, \tilde{M}_2, \mu_2, (C)^{-1}, \delta_2)$. The mapping $\varphi \to \varphi^*$ takes the set of homomorphisms φ for which $\varphi_*(C_1) \leqslant C_2$ onto the set of σ-homomorphisms $\alpha: M_2 \to M_1$ for which $\alpha^{(2)} \circ \mu_1 = \mu_2 \circ \alpha$. The mapping is one–one on equivalence classes of homomorphisms modulo agreement on conull sets.*

Proof. Any φ^* clearly satisfies the desired conditions. If α: $M_2 \to M_1$ is such a σ-homomorphism, let $f: G_1 \to G_2$ be any function with $f^* = \alpha$. Then $f(x)f(y)$ is defined and equal to $f(xy)$ for almost all $(x, y) \in G_1^{(2)}$, so by Theorem 5.1 of [14], there is a Borel function $\varphi: G_1 \to G_2$ which agrees with f a.e.(so $\varphi^* = \alpha$) and is algebraically a homomorphism on some i.c. of G_1. Denoting $\varphi \mid U_{G_1}$ by $\tilde{\varphi}$ we have $\tilde{\varphi} \circ d = d \circ \varphi$ a.e. so $\tilde{\varphi}_*(\tilde{C}_1) \leqslant \tilde{C}_2$ and hence φ is in fact a homomorphism. The last statement is true because $\varphi_1{}^* = \varphi_2{}^*$ iff $\varphi_1 = \varphi_2$ a.e.

The next result characterizes quotient measurable groupoids in Boolean terms.

THEOREM 4.2. *Let* $(M_1, \tilde{M}_1, \mu_1, (\ )^{-1}, \delta_1)$ *be a cogroupoid and let* M_2 *be a closed subalgebra of* M_1. *Set* $\tilde{M}_2 = \delta_1^{-1}(M_2)$, $\mu_2 = \mu_1 \mid M_2$ *let* $(\ )^{-1}$ *denote its own restriction to* M_2, *and set* $\delta_2 = \delta_1 \mid \tilde{M}_2$. *Then* $(M_2, \tilde{M}_2, \mu_2, (\ )^{-1}, \delta_2)$ *is a cogroupoid iff*

($q1$) $\{e^{-1}\colon e \in M_2\} = M_2$;

($q2$) $\mu_2(M_2)$ *is contained in the closed subalgebra of* $M_1^{(2)}$ *generated by* $j_1(M_2) \cup j_2(M_2)$, *where* j_1, j_2 *are the natural imbeddings of* M_1 *into* $M_1^{(2)}$.

If M_1 *is ergodic, so is* M_2.

Proof. First suppose $(M_2, \tilde{M}_2, \mu_2, (\ )^{-1}, \delta_2)$ is a cogroupoid. Let $(G_1, C_1)(G_2, C_2)$ be groupoids whose duals are the given cogroupoids. Then the inclusion of M_2 into M_1 induces a homomorphism φ of G_1 to G_2 such that $\varphi_*(C_1) = C_2$, by Theorem 4.1.

Proof of ($q1$). If $e \in M_2$, then there is a Borel $A \subseteq G_2$ such that $q(\varphi^{-1}(A)) = e$. Then $e^{-1} = q(\varphi^{-1}(A)^{-1}) = q(\varphi^{-1}(A^{-1})) \in M_2$.

Proof of ($q2$). We have

$$\begin{array}{ccc} G_1^{(2)} & \longrightarrow & G_2^{(2)} \\ \downarrow & & \downarrow \\ G_1 & \longrightarrow & G_2 \end{array}$$

a commutative diagram, so

$$\mu_1(M_2) = \mu_1(\varphi^*(M(C_2))) = \varphi^{(2)*}(\mu_2(M(C_2))) \subseteq \varphi^{(2)*}(M(C_2^{(2)})).$$

The latter is the σ-algebra generated by

$$\{\varphi^*(e_1) \otimes 1\colon e_1 \in M(C_2)\} \cup \{1 \otimes \varphi^*(e_2)\colon e_2 \in M(C_2)\}$$

according to Remark (a) after Theorem 1.6.

Now for the converse, suppose ($q1$) and ($q2$) hold. Then $\rho_1^{-1}(M_2) = \delta_1^{-1}(M_2) = \tilde{M}_2$ so $(\ )^{-1} \circ \delta_2 = \rho_1 \mid M_2$, and we denote this by ρ_2.

Represent M_2 as $M(\lambda)$ where λ is a measure on a standard space X and let $f\colon G_1 \to X$ be such that $f^*(M(\lambda)) = M_2 \subseteq M_1$, with f^* one–one. Represent $\tilde{M}_2$ as $M(\tilde{\lambda})$ where $\tilde{\lambda}$ is the image of λ on a quotient space Y of X, and let $g\colon U_{G_1} \to Y$ induce the inclusion of $M(\tilde{\lambda})$ into $\tilde{M}_1$. Then form $\lambda^{(2)}$ and $M_2^{(2)} = M(\lambda^{(2)})$. By Theorem 1.6, $f^{(2)} = f \times f \mid G_1^{(2)}$ induces an inclusion of $M_2^{(2)}$ into $M_1^{(2)}$ and the image is the closed algebra $M_2^{(2)\prime}$ generated by $j_1(M_2) \cup j_2(M_2)$. Clearly μ_2, δ_2 are imbeddings and $(\ )^{-1}$ is an automorphism of period 2. From $\mu_1 \circ \delta_1 = j_2 \circ \delta_1$ and $\mu_1 \circ \rho_1 = j_1 \circ \rho_1$ it follows that $\mu_2 \circ \delta_2 = j_2 \circ \delta_2$ and $\mu_2 \circ \rho_2 = j_1 \circ \rho_2$, and from $\mu_1{}^{+} \circ \mu_1 = \mu_{1+} \circ \mu_1$ it follows that $\mu_2{}^{+} \circ \mu_2 = \mu_{2+} \circ \mu_2$. Thus condition (a) of Definition 3.5 holds. For condition (b), note that if the automorphisms for M_1 are denoted τ_1, τ_2 then $\tau_1(j_1(M_2)) = \mu_1(M_2) \subseteq M_2^{(2)\prime}$ and $\tau_1(j_2(M_2)) = j_2((\ )^{-1}(M_2)) = j_2(M_2) \subseteq M_2^{(2)\prime}$, so that we must have $\tau_1(M_2^{(2)\prime}) \subseteq M_2^{(2)\prime}$, and since $\tau_1^{-1} = \tau_1$, τ_1 induces an automorphism of $M_2^{(2)\prime}$. Similarly τ_2 induces an automorphism of $M_2^{(2)\prime}$, and the composition properties required are inherited from M_1. Ergodicity is also automatic for M_2 if M_1 is ergodic.

It is not hard to prove directly that if G is a measurable groupoid then so is the equivalence relation $E = \{(r(x), d(x))\colon x \in G\}$ when regarded as a groupoid and that E is ergodic if G is. However we shall deduce this fact from Theorem 4.2 along with a description of the Boolean dual of E.

THEOREM 4.3. *Let (G, C) be a measurable groupoid with Boolean dual $B = (M, \tilde{M}, \mu, (\ )^{-1}, \delta)$. Let M_1 be the σ-algebra generated in M by $\delta(\tilde{M}) \cup \rho(\tilde{M})$ $(\rho = (\ )^{-1} \circ \delta)$. Then $B_1 = (M_1, \tilde{M}, \mu \mid M_1, (\ )^{-1}, \delta$ is a cogroupoid and is the Boolean dual of the measurable equivalence relation $E = \{(r(x), d(x))\colon x \in G\}$.*

Proof. Let m be the multiplication, $m\colon G^{(2)} \to G$, and let p_1, p_2 be the coordinate projections. Then $d \circ m = d \circ p_2$ so $\mu \circ \delta = j_2 \circ \delta$ and hence $\mu(\delta(\tilde{M})) \subseteq j_2(M_1)$. Similarly $\mu(\rho(\tilde{M})) \subseteq j_1(M_1)$, so $\mu(M_1)$ is contained in the σ-algebra generated by $j_1(M_1) \cup j_2(M_1)$; i.e., condition ($q2$) of Theorem 4.2 holds. Condition ($q1$) holds because $(\ )^{-1}(\delta(\tilde{M})) = \rho(\tilde{M})$.

Let (G_1, C_1) be a groupoid of which B_1 is the Boolean dual and let $g\colon G \to G_1$ be a homomorphism such that g^* is the imbedding of M_1 into M. The mappings δ, ρ take $\tilde{M}$ into M_1 and $g^* \circ \delta = \delta$, $g^* \circ \rho = \rho$, so we can define $d_1, r_1\colon G_1 \to U$, where U is the units of G, and have $d_1 \circ g = d$, a.e. and $r_1 \circ g = r$ a.e. Thus $(r, d)\colon G \to U \times U$ and $(r_1, d_1)\colon G_1 \to U \times U$ induce the same measure class on $U \times U$, i.e., $(r, d)_*(C) = (r_1, d_1)_*(C_1)$. Hence $(r_1, d_1)^*$ is an isomorphism, so (r_1, d_1) is essentially an isomorphism.

5. Boolean Duals of Homomorphism (II)

In this section we consider still another special class of homomorphisms to find a way to translate them into Boolean terms. Here we will not require that φ^* exist, but that $\tilde{\varphi}^*$ exists. Since the maps φ^* do not exist, then the φ's must be defined in a new way. We begin by giving an algebraic treatment of a different way to regard homomorphisms, then add the measure theoretic technicalities, and give the Boolean version only at the end. The idea is to connect homomorphisms with (a certain kind of) actions on the left.

Let G, H be groupoids with U_G, U_H their respective sets of units and let $\varphi\colon G \to H$ be a homomorphism. We will prove a theorem and then state it. If $\tilde{\varphi}(U_G) \neq U_H$, we can replace H by $H \mid \tilde{\varphi}(U_G)$, so we suppose $\tilde{\varphi}(U_G) = U_H$. Let $q = \tilde{\varphi}$. Then let $G \times' H = \{(x, y) \in G \times H\colon q \circ d(x) = d \circ \varphi(x) = r(y)\}$ and define $a\colon G \times' H \to H$ by $a(x, y) = \varphi(x)y$. Then $a(x, y_1 y_2) = a(x, y_1)y_2$ (both make sense and they are equal) whenever $(x, y_1, y_2) \in G \times' H^{(2)} = \{(x, y_1, y_2)\colon x \in G,\ (y_1, y_2) \in H^{(2)}$ and $q \circ d(x) = r(y_1)\}$. Also $a(x_1 x_2, y) = a(x_1, a(x_2, y))$ whenever $(x_1, x_2, y) \in G^{(2)} \times' H$, the latter set being defined similarly.

Conversely, let $q\colon U_G \to U_H$ be onto, define $G \times' H$, $G \times' H^{(2)}$, and $G^{(2)} \times' H$ as above, and suppose $a\colon G \times' H \to H$ is a function with those properties. Define $\varphi(x) = a(x, q \circ d(x))$: this makes sense because $r \circ q \circ d(x) = q \circ d(x)$. Now if $(x_1, x_2) \in G^{(2)}$ then $d(x_2) = d(x_1 x_2)$ and

$$\begin{aligned} \varphi(x_1 x_2) &= a(x_1 x_2, q \circ d(x_2)) \\ &= a(x_1, a(x_2, q \circ d(x_2))) \\ &= a(x_1, \varphi(x_2)). \end{aligned}$$

Since $a(x_1, \varphi(x_2))$ makes sense, $r(\varphi(x_2)) = q \circ d(x_1)$ and

$$\begin{aligned} \varphi(x_1 x_2) &= a(x_1, q \circ d(x_1)\, \varphi(x_2)) \\ &= a(x_1, q \circ d(x_1))\, \varphi(x_2) \\ &= \varphi(x_1)\, \varphi(x_2). \end{aligned}$$

Also if $(x, y) \in G \times' H$ then $a(x, y) = a(x, q \circ d(x)y) = \varphi(x)y$. Thus $\varphi \to (\tilde{\varphi}, a)$ is one–one and onto.

THEOREM 5.1. *If G and H are groupoids then there is a one–one correspondence between the set of homomorphisms φ: $G \to H$ with $\tilde{\varphi}$ onto and the set of pairs (q, a) for which q maps U_G onto U_H and a maps $G \times' H = \{(x, y) \in G \times H: q \circ d(x) = r(y)\}$ to H and the following are satisfied.*

$$\text{If } (x, y) \in G \times' H, \text{ then } r \circ a(x, y) = q \circ r(x). \tag{5.1}$$

$$\text{If } (x_1, x_2, y) \in G^{(2)} \times' H, \text{ then} \tag{5.2}$$

$$a(x_1 x_2, y) = a(x_1, a(x_2, y))$$

$$\text{If } (x, y) \in G \times' H, \text{ then } d \circ a(x, y) = d(y). \tag{5.3}$$

$$\text{If } (x, y_1, y_2) \in G \times' H^{(2)}, \text{ then} \tag{5.4}$$

$$a(x, y_1 y_2) = a(x, y_1) y_2 .$$

In case G and H are Borel groupoids, q and a are Borel iff φ is Borel.

Remark. Condition (5.1) guarantees that $a(x_1, a(x_2, y))$ makes sense in (5.2). The expression $a(x_1 x_2, y)$ automatically makes sense. Condition (5.3) guarantees that $a(x, y_1) y_2$ makes sense in (5.4). The expression $a(x, y_1 y_2)$ automatically makes sense.

Let $(G, [\lambda])$ and $(H, [\mu])$ be measurable groupoids and let φ be a Borel homomorphism of G to H for which $\tilde{\varphi}_*(\tilde{\lambda})$ and $\tilde{\mu}$ are equivalent. This corresponds to the algebraic condition that $\tilde{\varphi}$ take the set of units U_G in G onto the set of units U_H in H. We now want to supply the measure theoretic version of Theorem 5.1. After that we can convert the result into Boolean terms. The measure theoretic version requires measure classes on the fibered products $G \times' H$, $G^{(2)} \times' H$ and $G \times' H^{(2)}$ so that the appropriate identities can be required to hold only almost everywhere. It is essential for the Boolean version that we be able to work in the "almost everywhere" context.

To form the necessary fibered products of the measures, $\lambda \times' \mu$, $\lambda^{(2)} \times' \mu$ and $\lambda \times' \mu^{(2)}$, we first need some decompositions. Let q be any Borel function from U_G to U_H for which $q_*(\tilde{\lambda}) \sim \tilde{\mu}$ and set $p = q \circ d$. Then $p_*(\lambda) \sim \tilde{\mu}$. Let $\lambda = \int \lambda_u \, d\tilde{\lambda}(u)$ be a decomposition of λ relative to d and let $\tilde{\lambda} = \int \tilde{\lambda}_v \, d\tilde{\mu}(v)$ be a decomposition of $\tilde{\lambda}$ over $\tilde{\mu}$ relative to q. Then for $v \in U_H$, set $\lambda_{p,v} = \int \lambda_u \, d\tilde{\lambda}_v(u)$ and note that $\lambda = \int \lambda_{p,v} \, d\tilde{\mu}(v)$ is a decomposition of λ over $\tilde{\mu}$ relative to p. If $\pi(x_1, x_2) = p(x_2)$ for $(x_1, x_2) \in G^{(2)}$, then $\pi^{-1}(v) = \bigcup \{d^{-1}(r(x_2)) \times \{x_2\}: x_2 \in p^{-1}(v)\}$. For λ-almost every x_2 the measure $\lambda_{r(x_2)}$ is concentrated on $d^{-1}(r(x_2))$, so for $\tilde{\mu}$-almost every v the measure $\lambda^{(2)}_{\pi,v} = \int \lambda_{r(x)} \times \epsilon_x \, d\lambda_{p,v}(x)$ is concentrated on $\pi^{-1}(v)$, and

$$\begin{aligned}\int \lambda^{(2)}_{\pi,v} \, d\tilde{\mu}(v) &= \iint \lambda_{r(x)} \times \epsilon_x \, d\lambda_{p,v}(x) \, d\tilde{\mu}(v) \\ &= \int \lambda_{r(x)} \times \epsilon_x \, d\lambda(x) \\ &= \lambda^{(2)},\end{aligned}$$

so we have a decomposition of $\lambda^{(2)}$ over $\tilde{\mu}$ relative to π. Similarly the measures $\mu^{(2)v} = \int \epsilon_y \times \mu^{d(y)} \, d\mu^v(y)$ give a decomposition of $\mu^{(2)}$ over $\tilde{\mu}$ relative to $(y, z) \mapsto r(y)$, if $\mu = \int \mu^v \, d\tilde{\mu}(v)$ is a decomposition of μ over $\tilde{\mu}$ relative to r. By the general theory, the classes of

$$\lambda \times' \mu = \int \lambda_{p,v} \times \mu^v \, d\tilde{\mu}(v),$$

$$\lambda^{(2)} \times' \mu = \int \lambda^{(2)}_{\pi,v} \times \mu^v \, d\tilde{\mu}(v),$$

and

$$\lambda \times' \mu^{(2)} = \int \lambda_{p,v} \times \mu^{(2)v} \, d\tilde{\mu}(v)$$

depend only on $[\lambda]$ and $[\mu]$ and are concentrated on $G \times' H$, $G^{(2)} \times' H$ and $G \times' H^{(2)}$, respectively. These are the desired relative or fibered products.

The next lemma involves these measure classes. It will be needed to Booleanize homomorphisms as well as in the measure theoretic version of Theorem 5.1, and supplies the extra condition on functions a of Theorem 5.1 which is needed in the measure theoretic case.

LEMMA 5.2. *Let $\varphi: G \to H$ be a strict homomorphism for which $\tilde{\varphi}_*(\tilde{\lambda}) \sim \tilde{\mu}$ and define $q = \tilde{\varphi}$, $a(x, y) = \varphi(x) y$ for $(x, y) \in G \times' H$. Then*

$$a_*(\lambda \times' \mu) \ll \mu. \tag{5.5}$$

Proof. Let N be a μ-null Borel set and let $B = a^{-1}(N)$. We want to show that B is $\lambda_{p,v} \times \lambda^v$-null for $\tilde{\mu}$-almost every $v \in U_H$. Thus for purposes of the proof, take an i.c., if necessary, to guarantee that $y \cdot \mu^{d(y)} \sim \mu^{r(y)}$ for $y \in H$. Then whenever $p(x) = v$ we have $\{y: (x, y) \in B\} = \{y: r(y) = v \text{ and } \varphi(x) y \in N\}$ a μ^v-null set if N is

$\varphi(x) \cdot \mu^{p(x)}$-null (recall $p(x) = d \circ \varphi(x)$), i.e., if N is $\mu^{r\circ\varphi(x)} = \mu^{p(x^{-1})}$-null. Now $\{v \in U_H: \mu^v(N) = 0\}$ is $\tilde{\mu}$-conull so $\{u \in U_G: \mu^{q(u)}(N) = 0\}$ is $\tilde{\lambda}$-conull and hence $A = \{x \in G: \mu^{p(x)}(N) = 0\}$ is λ-conull. Thus $\{v \in U_H: A \text{ is } \lambda_{p,v}\text{-conull}\}$ is $\tilde{\mu}$-conull. If $x^{-1} \in A$ and $p(x) = v$, then $\{y: (x, y) \in B\}$ is μ^v-null, so if A is $\lambda_{p,v}$-conull then B is $\lambda_{p,v} \times \mu^v$-null, by the Fubini Theorem. Hence B is $\lambda_{p,v} \times \mu^v$-null for $\tilde{\mu}$-almost all v, as desired.

THEOREM 5.3. *Let $(G, [\lambda])$, $(H, [\mu])$ be analytic measurable groupoids. Let $\mathscr{H}$ denote the set of homomorphisms φ from G to H for which $\tilde{\varphi}_*(\tilde{\lambda}) \sim \tilde{\mu}$ and let $\mathscr{A}$ denote the set of pairs (q, a) where $q: U_G \to U_H$ is a Borel function with $q_*(\tilde{\lambda}) \sim \tilde{\mu}$ and a is a Borel function from $G \times' H \to H$ such that (5.1) to (5.4) are satisfied almost everywhere and (5.5) holds. If $a_\varphi(x, y) = \varphi(x)y$ whenever $d \circ \varphi(x) = r(y)$ then $\varphi \mapsto (\tilde{\varphi}, a_\varphi)$ takes $\mathscr{H}$ into $\mathscr{A}$. If furthermore φ_1 and φ_2 are identified when $\varphi_1 = \varphi_2$ a.e. and a suitable similar identification is done for $\mathscr{A}$, then this mapping is one–one and onto.*

Proof. We know the function takes $\mathscr{H}$ into $\mathscr{A}$. If $\varphi_1, \varphi_2 \in \mathscr{H}$ and $\varphi_1 = \varphi_2$ a.e., then φ_1 and φ_2 agree on some i.c. of G, by Lemma 5.2 of [14], because the set where they agree is a conull subset closed under multiplication. Hence $\tilde{\varphi}_1 = \tilde{\varphi}_2$ a.e. Now whenever $q_1 = q_2$ a.e. we can choose a conull Borel set $U_1 \subseteq U_G$ on which q_1 and q_2 agree. If $V_1 = q_1(U_1)$ then V_1 is a conull analytic set in U_H and hence contains a conull Borel set V_2. If we replace U_1 by $U_1 \cap q_1^{-1}(V_2)$ we reduce to the case in which V_1 itself is Borel. Then the spaces $(G \mid U_1) \times' (H \mid V_1)$ are the same whether q_1 or q_2 is used in defining this relative product, since $q_1 = q_2$ on U_1. Thus the two spaces $G \times' H$ have conull subsets which are actually the same set. Then it makes sense to say that $a_1 = a_2$ a.e. even though their domains are not the same. In particular when $\varphi_1 = \varphi_2$ a.e. we can choose U_1 so that φ_1 and φ_2 agree on $G \mid U_1$ and hence $a_{\varphi_1} = a_{\varphi_2}$ on $(G \mid U_1) \times' (H \mid V_1)$ so that $a_{\varphi_1} = a_{\varphi_2}$ a.e.

If $\tilde{\varphi} = \tilde{\psi}$ a.e., and $a_\varphi = a_\psi$ a.e. then for almost every y in H we have $\varphi(x)y = \psi(x)y$ (and hence $\varphi(x) = \psi(x)$) for $\lambda_{p,r(y)}$-almost all x in $p^{-1}(r(y))$, when $p = \varphi \circ d$. Hence $\varphi(x) = \psi(x)$ for λ-almost every x in G. Thus $\varphi \mapsto (\tilde{\varphi}, a_\varphi)$ is one–one on the equivalence class level.

To show that the mapping is onto, start with a pair $(q, a) \in \mathscr{A}$. Let L_1 be the conull Borel set $\{(x, y_1, y_2) \in G \times' H^{(2)}: d \circ a(x, y_1) = d(y_1)$ and $a(x, y_1y_2) = a(x, y_1)y_2\}$ and let L_2 be the conull Borel set $\{(x_1, x_2, y) \in G^{(2)} \times' H: r \circ a(x_2, y) = q \circ r(x_2)$ and $a(x_1x_2, y) =$

$a(x_1, a(x_2, y))\}$. Again take $p = q \circ d$, denote point masses by ϵ's and take decompositions of measures as above. Let

$$\begin{aligned} Y &= \{y \in H: a(x, yz) = a(x, y)z \text{ for } \lambda_{p,r(y)} \times \mu^{d(y)}\text{-almost all pairs } (x, z)\} \\ &= \{y \in H: \lambda_{p,r(y)} \times \epsilon_y \times \mu^{d(y)}(G \times' H^{(2)} - L_1) = 0\}. \end{aligned}$$

Since L_1 is conull, it is $\lambda_{p,v} \times \mu^{(2)v}$-conull for $\tilde{\mu}$-almost every v. Thus for almost every v we know L_1 is $\lambda_{p,v} \times \epsilon_y \times \mu^{d(y)}$-conull for μ^v-almost every y. This implies that L_1 is $\lambda_{p,r(y)} \times \epsilon_y \times \mu^{d(y)}$-conull for μ-almost every y in H, i.e., that Y is conull. Hence $r(Y)$ is a $\tilde{\mu}$-conull analytic set in U_H. It follows from the von Neumann Selection Lemma that there is a Borel function g: $U_H \to H$ such that $r \circ g$ agrees with the identity on U_H almost everywhere and $g^{-1}(Y)$ is $\tilde{\mu}$-conull. Choose a conull Borel set V with $r \circ g(v) = v$ for $v \in V$ and $g(V) \subseteq Y$, and so that $y \cdot \mu^{d(y)} \sim \mu^{r(y)}$ if $y \in H \mid V$.

Now define a_1: $G \times' H \to H$ as follows: if $p(x) = r(z) = v \in V$, set $a_1(x, z) = a(x, y)y^{-1}z$ where $y = g(v)$, and otherwise let $a_1(x, z) = a(x, z)$. If $v \in V$ and $y = g(v)$, then $z \mapsto y^{-1}z$ is a null-set-preserving Borel isomorphism of $r^{-1}(v)$ onto $r^{-1}(d(y))$, so

$$\begin{aligned} a_1(x, z) &= a(x, y)y^{-1}z \\ &= a(x, y(y^{-1}z)) \\ &= a(x, z) \end{aligned}$$

for $\lambda_{p,v} \times \mu^v$-almost all (x, z). Thus $a_1 = a$ a.e. relative to $\lambda \times' \mu$. Furthermore, if $r(z_1) = v$ and $r(z_2) = d(z_1)$, then for any $x \in p^{-1}(v)$ we have

$$\begin{aligned} a_1(x, z_1z_2) &= a(x, y)(y^{-1}z_1z_2) \\ &= (a(x, y)(y^{-1}z_1))z_2 \\ &= a_1(x, z_1)z_2 . \end{aligned}$$

In particular, if $G_1 = G \mid q^{-1}(V)$ and $H_1 = H \mid V$ then $a_1(x, z_1z_2) = a_1(x, z_1)z_2$ for (x, z_1, z_2) in $G_1 \times' H_1^{(2)}$.

Next we show that a_1 satisfies (5.2) a.e. If $N = \{(x, z): a_1(x, z) \neq a(x, z)\}$, then N is $\lambda \times' \mu$-null, so for λ-almost every x we have $\mu^{p(x)}(\{z: (x, z) \in N\}) = 0$. Now $(x_1, x_2) \mapsto x_1x_2$ pulls null sets back to null sets, and $p(x_1x_2) = p(x_2)$, so $\mu^{p(x_2)}(\{z: (x_1x_2, z) \in N\}) = 0$ for $\lambda^{(2)}$-almost all (x_1, x_2). Thus $a_1(x_1x_2, z) = a(x_1x_2, z)$ for $\lambda^{(2)} \times' \mu$ almost all (x_1, x_2, z). Now a and a_1 pull back null sets to null sets according to (5.5), so a similar argument shows that $a_1(x_1, a_1(x_2, z)) =$

$a(x_1, a(x_2, z))$ a.e. relative to $\lambda^{(2)} \times' \mu$. These two almost everywhere statements together with the fact that a satisfies (5.2) a.e. combine to show that a_1 satisfies (5.2) a.e.

Thus $A = \{(x_1, x_2, z) \in G_1^{(2)} \times' H_1 : a_1(x_1x_2, z) = a_1(x_1, a_1(x_2, z))\}$ is a conull set, so for almost every $(x_1, x_2) \in G^{(2)}$ there is a $z \in H$ such that $(x_1, x_2, z) \in A$. For any such triple $a_1(x_1, a_1(x_2, z))$ makes sense so $p(x_1) = r(a_1(x_2, z))$, and $p(x_1x_2) = p(x_2) = r(z) = zz^{-1}$. Also $x_1, x_2, x_1x_2 \in G_1$ and $z, z^{-1} \in H_1$, so $a_1(x_2, z)z^{-1}$ makes sense and hence $a_1(x_2, z) \in H_1$. In fact we have proved that $a_1(G_1 \times' H_1) \subseteq H_1$. Thus the following calculation is correct for $(x_1, x_2, z) \in A$:

$$\begin{aligned} a_1(x_1, p(x_1))\, a(x_2, p(x_2)) = a_1(x_1, a_1(x_2, p(x_2))) &= a_1(x_1, a_1(x_2, z)z^{-1}) \\ &= a_1(x_1, a_1(x_2, z))z^{-1} \\ &= a_1(x_1x_2, z)z^{-1} \\ &= a_1(x_1x_2, p(x_1x_2)). \end{aligned}$$

If we define $\varphi_1(x) = a_1(x, p(x))$ for $x \in G_1$, $\varphi_1(x) = p(x)$ if $x \notin G_1$, the preceding argument shows that for almost every $(x_1, x_2) \in G_1^{(2)}$ (and hence for almost every $(x_1, x_2) \in G^{(2)}$) the product $\varphi_1(x_1)\, \varphi_1(x_2)$ is defined and equal to $\varphi_1(x_1x_2)$. In addition, $a_1(x, z) = \varphi_1(x)z$ for $(x, z) \in G_1 \times' H_1$. Now let φ be a homomorphism which agrees a.e. with φ_1. Then for almost all $(x, z) \in G_1 \times' H_1$ we have $\varphi(x)z$ defined (and equal to $\varphi_1(x)z$) so that $d(\varphi(x)) = r(z)$, i.e., $\tilde{\varphi} \circ d(x) = r(z)$. It follows that $\tilde{\varphi} = q$ a.e., and we also have $a_\varphi = a_1$ a.e. from above.

Now to convert all this into Boolean terms we need to have conditions (5.1) to (5.4) given as function equalities, which then translate into σ-homomorphism equalities. To do this, first let m denote the multiplication in either G or H, let p_1 denote the projection of $G \times' H$ onto G, let p_2 denote the projection of $G \times' H$ onto H, let p_3 denote the projection of $G^{(2)}$ onto G by $p_3(x_1, x_2) = x_2$ and let p_4 denote the projection of $H^{(2)}$ onto H by $p_4(y_1, y_2) = y_1$. We automatically have $d \circ p_3 = d \circ m$ and $r \circ p_4 = r \circ m$, so $m \times i$ takes $G^{(2)} \times' H$ into $G \times' H$ and $i \times m$ takes $G \times' H^{(2)}$ into $G \times' H$. Condition (5.1) is equivalent to

$$r \circ a = q \circ r \circ p_1, \tag{5.1f}$$

which is also equivalent to having $i \times a$ take $G \times' (G \times' H) = G^{(2)} \times' H$ into $G \times' H$. Condition (5.2) then is equivalent to

$$a \circ (m \times i) = a \circ (i \times a) \text{ on } G^{(2)} \times' H. \tag{5.2f}$$

Next, condition (5.3) is equivalent to

$$d \circ p_2 = d \circ a \tag{5.3f}$$

which is equivalent to having $a \times i$ map $(G \times' H) \times' H = G \times' H^{(2)}$ into $H^{(2)}$. Finally condition (5.4) is equivalent to

$$a \circ (i \times m) = m \circ (a \times i) \text{ on } G \times' H^{(2)}. \tag{5.4f}$$

The point of condition 5.5 is that without it the function a: $G \times' H \to H$ would not have a Boolean dual at all.

Now for the Boolean version we start with an imbedding γ: $M(\tilde{\mu}) \to M(\tilde{\lambda})$ and form the relative tensor product $M(\lambda) \otimes' M(\mu)$, using $\delta \circ \gamma$: $M(\tilde{\mu}) \to M(\lambda)$ and ρ: $M(\tilde{\mu}) \to M(\mu)$. (Recall that we use δ and ρ for the domain and range σ-homomorphisms.) Then we need a σ-homomorphism α: $M(\mu) \to M(\lambda) \otimes' M(\mu)$, so that the Boolean duals of (5.1f) to (5.4f) hold. Let us use m^* to denote the multiplication in the Boolean dual of G and of H, B_G and B_H respectively, let j_1 denote the injection of $M(\lambda)$ into $M(\lambda) \otimes' M(\mu)$, let j_2 denote the injection of $M(\mu)$ into $M(\lambda) \otimes' M(\mu)$, let j_3 denote the second injection of $M(\lambda)$ into $M(\lambda^{(2)}) = M(\lambda) \otimes' M(\lambda)$, and let j_4 denote the first injection of $M(\mu)$ into $M(\mu^{(2)}) = M(\mu) \otimes' M(\mu)$.

As in the case of products of sets there are natural isomorphisms between $M(\lambda^{(2)}) \otimes' M(\mu)$ (relative product over $j_3 \circ \delta \circ \gamma$ and ρ) and $M(\lambda) \otimes' (M(\lambda) \otimes' M(\mu))$ (relative product over δ and $j_1 \circ \rho$), and we choose to ignore the difference, just as we did with $G^{(2)} \times' H$ and $G \times' (G \times' H)$. A similar identification will be made between $M(\lambda) \otimes' M(\mu^{(2)})$ and $(M(\lambda) \otimes' M(\mu)) \otimes' M(\mu)$. We think of which ever model is convenient for given maps under consideration.

THEOREM 5.4. *Let $(G, [\lambda])$ and $(H, [\mu])$ be analytic measurable groupoids. Let $\mathscr{H}_1$ denote the set of Borel homomorphisms φ: $G \to H$ such that $\tilde{\varphi}_*(\tilde{\lambda}) \sim \tilde{\mu}$, and let $\mathscr{H}_2$ denote the set of pairs (γ, α) for which γ is an imbedding of $M(\tilde{\mu})$ into $M(\tilde{\lambda})$ and α is a σ-homomorphism of $M(\mu)$ into $M(\lambda) \otimes' M(\mu)$ and these four conditions hold*:

$$\alpha \circ \rho = j_1 \circ \rho \circ \gamma \tag{5.1b}$$

$$(m^* \otimes' i) \circ \alpha = (i \otimes' \alpha) \circ \alpha \tag{5.2b}$$

$$j_2 \circ \delta = \alpha \circ \delta \tag{5.3b}$$

$$(i \otimes' m^*) \circ \alpha = (\alpha \otimes' i) \circ m^*. \tag{5.4b}$$

Then $\varphi \to (\tilde{\varphi}^*, a_\varphi{}^*)$ maps $\mathscr{H}_1$ onto $\mathscr{H}_2$ and φ_1, φ_2 have the same image iff $\varphi_1 = \varphi_2$ a.e.

Proof. From (5.1b)t ogether with an automatic condition we see that $m^* \otimes' i$ and $i \otimes' \alpha$ exist, taking $M(\lambda) \otimes' M(\mu)$ to $M(\lambda^{(2)}) \otimes' M(\mu)$ and $M(\lambda) \otimes' (M(\lambda) \otimes' M(\mu))$ respectively, and the equality asserted in (5.2b) is intended to include the identification. A similar explanation goes with (5.3b) and (5.4b). Otherwise, the proof was given in Theorem 5.3, and in fact the "onto" part is the only thing not obvious. If q and a are any functions with $q^* = \gamma$ and $a^* = \alpha$, and we let $j_k = p_k{}^*$ where p_k is the actual projection in each case, then (5.1b) to (5.4b) imply that (5.1) to (5.4) hold almost everywhere. Hence there is a φ in $\mathscr{H}_1$ with $\tilde{\varphi}^* = \gamma$, $a_\varphi{}^* = \alpha$.

6. Quasiinvariant Decompositions

In working with measurable groupoids, the invariance of the measure class is very important, and at times a slight strengthening of that property would be convenient. In this section we point out that the desired property holds for groupoids arising from actions of groups, and can be achieved in general by passing to an i.c. We also illustrate the usefulness with an application.

Definition 6.1. Let F be an analytic Borel groupoid and let λ be a finite measure on F. If $\tilde{\lambda} = d_*(\lambda)$ and $\lambda = \int \lambda_u \, d\tilde{\lambda}(u)$ is a decomposition of λ over $\tilde{\lambda}$ such that $\lambda_{r(x)}x \sim \lambda_{d(x)}$ for every $x \in F$, then we shall say the decomposition is *right quasiinvariant*. The term *left quasiinvariant* is defined similarly.

The proof of Theorem 4.3 of [14] shows that if S is a G-space μ a finite measure on S and ν is a probability measure in the Haar class on G then $\mu \times \nu$ has a left quasiinvariant decomposition.

Lemma 6.2. *Let G be an analytic Borel groupoid and let λ be a probability measure on G such that $[\lambda]$ is invariant. If $\lambda = \int \lambda_u \, d\tilde{\lambda}(u)$ is any decomposition of λ relative to d, then there is an inessential contraction G_0 of G such that if $x \in G_0$, then $\lambda_{r(x)} \cdot x \sim \lambda_{d(x)}$ and $\lambda_{r(x)}(G_0) = \lambda_{r(x)}(d^{-1}(r(x))) = \lambda_{r(x)}(G) = 1$. Thus every measurable groupoid has an inessential contraction on which the restricted measure has a right quasiinvariant decomposition.*

Proof. First choose a conull Borel set $U_1 \subseteq U$ such that if $u \in U_1$ then $\lambda_u(d^{-1}(u)) = \lambda_u(G) = 1$ and if $x \in G \mid U_1$ then $\lambda_{r(x)} \cdot x \sim \lambda_{d(x)}$.

Since $G \mid U_1$ is conull, there is a conull Borel set $U_2 \subseteq U_1$ such that $u \in U_2$ implies $\lambda_u(G \mid U_1) = 1$. We continue by induction: if $U_1 \supseteq \cdots \supseteq U_n$ are Borel and U_n is conull, then there is a conull Borel set $U_{n+1} \subseteq U_n$ such that $u \in U_{n+1}$ implies $\lambda_u(G \mid U_n) = 1$. Set $U_0 = \bigcap_{n=1}^{\infty} U_n$. Then if $u \in U_0$ and $n \geqslant 1$ we have $\lambda_u(G \mid U_n) = 1$. Hence if $u \in U_0$ we have $\lambda_u(G \mid U_0) = 1$. Since $U_0 \subseteq U_1$, if $x \in G \mid U_0$ then $\lambda_{r(x)} \cdot x \sim \lambda_{d(x)}$. Since $G_0 = G \mid U_0$ is conull for λ and every λ_u for $u \in U_0$, we may induce measures on G_0 by restriction, producing thereby a measure with a quasiinvariant decomposition.

LEMMA 6.3. *Let F be a groupoid, and suppose there is a measure λ such that $[\lambda]$ is invariant and λ has a right quasiinvariant decomposition. Then for every Borel $A \subseteq U$ with $\lambda(r^{-1}(A) + d^{-1}(A)) = 0$ there is a saturated Borel $A_1 \subseteq U$ such that $A + A_1$ is null.*

Proof. Suppose $\lambda = \int \lambda_u \, d\tilde{\lambda}(u)$ is a quasiinvariant decomposition and for $u \in U$ define $\nu_u = r_*(\lambda_u)$. Then $u \sim v$ implies $\nu_u \sim \nu_v$, since if $u = r(x)$ and $v = d(x)$, $\lambda_{r(x)}(r^{-1}(B)) = \lambda_{r(x)}(r^{-1}(B) \cap d^{-1}(r(x))) = \lambda_{r(x)}(\{y\colon d(y) = r(x) \text{ and } r(yx) = r(y) \in B\}) = (\lambda_{r(x)} \cdot x)(r^{-1}(B)) = 0$ iff $\lambda_{d(x)}(r^{-1}(B)) = 0$. Also $\tilde{\lambda} \sim \int \nu_u \, d\tilde{\lambda}(u)$ (and $u \mapsto \nu_u(A)$ is Borel for Borel A). Thus if A is given, the set $A_1 = \{u \in U\colon \nu_u(U - A) = 0\}$ is Borel and $u \sim v \in A_1$ implies $u \in A_1$, so A_1 is saturated. To show that $A_1 + A$ is null, note that $\lambda(r^{-1}(A) + d^{-1}(A)) = 0$ implies that $\lambda_u(r^{-1}(A) + d^{-1}(A)) = 0$ for almost all u. Hence $\nu_u(A) = \lambda_u(r^{-1}(A)) = \lambda_u(d^{-1}(A))$ for $\tilde{\lambda}$-almost all u. Thus $\nu_u(U - A) = 0$ for almost all $u \in A$ and is >0 for almost all $u \notin A$, which easily gives the desired conclusion.

COROLLARY 6.4. *For such (F, λ), $(F, [\lambda])$ is ergodic iff every saturated Borel set in $U = U_F$ is null or conull.*

COROLLARY 6.5 (Mackey [7]). *If (S, μ) is an analytic G-space with quasiinvariant μ, then μ is ergodic for the action of G iff every invariant Borel set in U is null or conull.*

Remarks. (a) Mackey deduced Corollary 6.5 from a uniqueness theorem for point actions giving rise to given Boolean actions. His proof depended on separability. Our proof seems to apply whenever S is a "good" space, μ is a finite measure on S and G is σ-compact so that there is a finite measure equivalent to Haar measure on G. It is possible that the correspondence between point actions and Boolean actions also holds for σ-compact groups.

(b) It should be possible to simplify the proof of Theorem 7.11 of [14] by using this Lemma 4.3. At least the analytic sets can be replaced by Borel sets.

LEMMA 6.6. *Let* $(G, [\lambda])$ *be a measurable groupoid with* $\lambda(G) = 1$ *and* $\lambda = \lambda^{-1}$. *Let* $E = (r, d)(G)$ *be the associated equivalence relation on* $U = U_G$ *and set* $\lambda' = (r, d)_*(\lambda)$. *Let* $\lambda' = \int \lambda_u' \, d\tilde{\lambda}(u)$ *be a decomposition of* λ' *relative to* d; *let* $\lambda = \int \lambda_{u,v} \, d\lambda'(u, v)$ *be a decomposition of* λ *relative to* (r, d); *for* $u \in U$ *let* $\lambda_u'' = r_*(\lambda_u')$ *and set* $\lambda_u = \int \lambda_{v,u} \, d\lambda_u''(v)$. *Then* $\lambda = \int \lambda_u \, d\tilde{\lambda}(u)$ *is a decomposition of* λ *relative to* d *and there is a conull Borel set* $U_0 \subseteq U$ *such that if* $G_0 = G \mid U_0$ *and* $E_0 = E \mid U_0$, *then*:

(1) $u \in U_0$ *implies* $\lambda_u'(E_0) = \lambda_u'([u] \times \{u\}) = 1$ (*recall* $[u] = \{v\colon v \sim u\}$);

(2) $(u, v) \in E_0$ *implies* $\lambda_u' \cdot (u, v) \sim \lambda_v'$;

(3) $u \in U_0$ *implies* $\lambda_u(G_0) = \lambda_u(d^{-1}(u)) = 1$;

(4) $x \in G_0$ *implies* $\lambda_{r(x)} \cdot x \sim \lambda_{d(x)}$.

Also, the $\lambda_{u,v}$ *may be chosen so that* $\lambda_{u,v} = (\lambda_{v,u})^{-1}$, *and if* $\lambda^u = (\lambda_u)^{-1}$, *then* $\int \lambda^u \, d\tilde{\lambda}(u)$ *is a left quasiinvariant decomposition of* λ.

Proof. The first conclusion follows from the first conclusion of Lemma 1.2. Since λ is symmetric, $\lambda_{u,v} = (\lambda_{v,u})^{-1}$ for λ'-almost all (u, v), so the measures $(1/2)(\lambda_{u,v} + (\lambda_{v,u})^{-1})$ also give a decomposition of λ over λ' with the desired symmetry property. By Lemma 6.2, we can choose conull Borel sets $U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots$ in U such that $E \mid U_{2n+1}$ satisfies (1) and (2) for $n = 0, 1, 2,\ldots$, and $G \mid U_{2n}$ satisfies (3) and (4) for $n = 1, 2,\ldots$. If $U_0 = \bigcap \{U_n\colon n \geqslant 1\}$ then (1), (2), (3), and (4) hold.

LEMMA 6.7. *With hypotheses as in Lemma* 6.6, *there is a Borel function* $(u, v) \to \lambda_{u,v}^*$ *from* E *to measures on* G *such that* $\lambda_{u,v}^* \sim \lambda_{u,v}$ *for* λ'-*almost all* (u, v) *and a conull Borel set* $U_0 \subseteq U$ *such that if* $u \in U_0$, $z \in G \mid U_0$ *and* $r(z) \sim u$, *then* $\lambda_{u,r(z)}^* \cdot z \sim \lambda_{u,d(z)}^*$ *and* $z \cdot \lambda_{d(z),u}^* \sim \lambda_{r(z),u}^*$. *Almost every* $\lambda_{u,v}^*$ *is a probability.*

Proof. Without loss of generality we may suppose that (1)–(4) of Lemma 6.6 hold for G, E, and $\lambda_{u,v} = (\lambda_{v,u})^{-1}$ always. Then if $(u, v) \in E$, $\lambda_u' \cdot (u, v) \sim \lambda_v'$ so $\lambda_u'' = r_*(\lambda_u') = r_*(\lambda_u' \cdot (u, v)) \sim r_*(\lambda_v') = \lambda_v''$. Hence for $x \in G$, $\lambda_{d(x)} = \int \lambda_{u,d(x)} \, d\lambda_{d(x)}''(u) \sim \int \lambda_{u,d(x)} \, d\lambda_{r(x)}''(u)$. Since $\lambda_{r(x)} \cdot x = \int \lambda_{u,r(x)} \cdot x \, d\lambda_{r(x)}''(u)$, by Lemma 1.2 and the fact that the $\lambda_{u,v}$ decompose λ over λ', the set $F = \{x \in G\colon \lambda_{u,r(x)} \cdot x \sim \lambda_{u,d(x)}$ and

$\lambda_{u,d(x)}$ is a measure concentrated on $r^{-1}(u) \cap d^{-1}(d(x))$, for $\lambda''_{r(x)}$-almost every $u\}$ is a conull set in G. Now if $(x, y) \in G^{(2)} \cap F \times F$, then $r(x) \sim r(y)$ so $\lambda''_{r(xy)} = \lambda''_{r(x)} \sim \lambda''_{r(y)}$; also $\lambda_{u,d(x)} = \lambda_{u,r(y)}$ since $r(y) = d(x)$. Hence

$$\begin{aligned}\lambda_{u,r(xy)} \cdot (xy) &= (\lambda_{u,r(x)} \cdot x) \cdot y \\ &\sim (\lambda_{u,r(y)}) \cdot y \\ &\sim \lambda_{u,d(y)} \\ &= \lambda_{u,d(xy)} \,.\end{aligned}$$

Thus $xy \in F$. By Lemma 5.2 of [14], F contains an i.c. $G_0 = G \mid U_0$. For $x \in G_0$, $\lambda_{u,r(x)} \cdot x \sim \lambda_{u,d(x)}$ and $\lambda_{u,r(x)}$ is concentrated on $r^{-1}(u) \cap d^{-1}(r(x))$ for $\lambda''_{r(x)}$-almost every u. For any Borel set $A \subseteq G$ and any x, u with $d(x) \sim u$, remembering that $By = \{by\colon (b, y) \in G^{(2)}$ and $b \in B\}$, we have

$$\begin{aligned}(\lambda_{u,r(x)} \cdot x)(A^{-1}) &= \lambda_{u,r(x)}(A^{-1}x^{-1}) \\ &= \lambda_{u,r(x)}((xA)^{-1}) \\ &= \lambda_{r(x),u}(xA) \\ &= x^{-1}\lambda_{d(x^{-1}),u}(A).\end{aligned}$$

Thus for $x \in G_0$, using symmetry of the $\lambda_{u,v}$, we have $x \cdot \lambda_{d(x),u} \sim \lambda_{r(x),u}$ for $\lambda''_{r(x)}$-almost all u. We may also, by a decreasing sequence again, choose U_0 so that (1)–(4) still hold for G_0, E_0, and hence may as well suppose $G_0 = G$, or that these invariance properties hold on G.

For $v \in U$, let $A_v = \{(u, x) \in [v] \times d^{-1}(v)\colon \lambda_{u,r(x)} \cdot x \sim \lambda_{u,v}\}$. If $d(x) = v$, then $\{u\colon (u, x) \in A_v\}$ is $\lambda''_{r(x)}$-conull and hence λ''_v-conull. By the Fubini Theorem, $\{u\colon \lambda_v(\{x\colon (u, x) \in A_v\}) = 1\}$ is λ''_v-conull.

Now set $T = \{(x, y) \in G^{(2)}\colon x \cdot (\lambda_{d(x),r(y)} \cdot y) \sim \lambda_{r(x),d(y)}\}$. By Lemma 2.1, the two functions from $G^{(2)}$ to $\mathscr{M}(G)$ are Borel functions, and since the set of pairs of equivalent measures is a Borel set, T is a Borel set in $G^{(2)}$. We will show that for λ'-almost every (u, v) the set T is $\lambda^u \times \lambda_v$-conull. Let B be the set of such pairs; $B \subseteq E$ and B is Borel. First notice that if $u \in U$ then $V_u = \{v \in U\colon \{x\colon x\lambda_{d(x),v} \sim \lambda_{r(x),v}\}$ is λ^u-conull$\}$ is λ''_u-conull. If $v \in V_u$ and $d(y) = v$, then $\{w\colon \lambda_{w,r(y)} \cdot y \sim \lambda_{w,v}\}$ is λ''_v-conull. Since $\lambda''_v \sim d^*(\lambda^u)$, the set $\{x \in r^{-1}(u)\colon \lambda_{d(x),r(y)} \cdot y \sim \lambda_{d(x),v}\}$ is λ^u-conull. Thus if $u \in U$, $v \in V_u$ and $d(y) = v$, we have $x \cdot (\lambda_{d(x),r(y)} \cdot y) \sim x \cdot (\lambda_{d(x),v}) \sim \lambda_{u,v} =$

$\lambda_{r(x),d(y)}$ for λ^u-almost all x. Hence for any $u \in U$, if $v \in V_u$ then $(u, v) \in B$, so $\lambda'(E - B) = 0$.

For $(u, v) \in E$, define

$$\lambda^*_{u,v} = \int x \cdot (\lambda_{d(x),r(y)} \cdot y)\, d\lambda^u \times \lambda_v(x, y).$$

Lemma 2.8 guarantees this is a Borel function of (u, v); and we have $\lambda^*_{u,v} \sim \lambda_{u,v}$ if $(u, v) \in B$. Also set $\lambda^+_{u,v} = \int \lambda_{u,r(y)} \cdot y\, d\lambda_v(y)$. If $u \in U$ and $z \in G$ with $r(z) \sim u$, we can calculate, using $\lambda_{r(z)} \cdot z \sim \lambda_{d(z)}$ near the end, that

$$\begin{aligned} \lambda^+_{u,r(z)} \cdot z &= \int (\lambda_{u,r(y)} \cdot y) \cdot z\, d\lambda_{r(z)}(y) \\ &= \int \lambda_{u,r(y)} \cdot (yz)\, d\lambda_{r(z)}(y) \\ &= \int \lambda_{u,r(y)} \cdot y\, d\lambda_{r(z)}(yz^{-1}) \\ &\sim \int \lambda_{u,r(y)} \cdot y\, d\lambda_{d(z)}(y) \\ &= \lambda^+_{u,d(z)}\,. \end{aligned}$$

Now

$$\lambda^*_{u,v} = \int x \cdot \lambda^+_{d(x),v}\, d\lambda^u(x), \qquad \text{so}$$

$$\begin{aligned} \lambda^*_{u,r(z)} \cdot z &= \int x \cdot (\lambda^+_{d(x),r(z)} \cdot z)\, d\lambda^u(x) \\ &\sim \int x \cdot \lambda^+_{d(x),d(z)}\, d\lambda^u(x) \\ &= \lambda^*_{u,d(z)}\,. \end{aligned}$$

This holds for any such u, z, and by a symmetric argument we have $z \cdot \lambda^*_{d(z),v} \sim \lambda^*_{r(z),v}$ for all z, v for which the statement makes sense.

THEOREM 6.8. *Let $(G, [\lambda])$ be a measurable groupoid. Then there is an i.c. $G \mid U_0$ of G and a measure $\lambda^* \sim \lambda$ with a decomposition over E such that*

(a) *for $(u, v) \in E \mid U_0$, $\lambda^*_{u,v}$ is a probability measure concentrated on $r^{-1}(u) \cap d^{-1}(v)$.*

(b) *for $(u, v) \in E \mid U_0$, $(\lambda^*_{u,v})^{-1} = \lambda^*_{v,u}$*

(c) *if* $x \in G \mid U_0$, $u \in U_0$ *and* $r(x) \sim u$, *then*

$$\lambda_{u,r(x)} \cdot x \sim \lambda_{u,d(x)}$$

and

$$x \cdot \lambda_{d(x),u} \sim \lambda_{r(x),u} \,.$$

Proof. We may suppose λ is a probability measure with $\lambda = \lambda^{-1}$. Take $\lambda^* = \int \lambda^*_{u,v}\, d\lambda'(u, v)$ where the $\lambda^*_{u,v}$ are provided by Lemma 6.7. Since $\lambda^*_{u,v} \sim \lambda_{u,v}$ for λ'-almost every (u, v), $\lambda^* \sim \lambda$. Properties (b) and (c) hold for some U_0, and $\lambda^*_{u,v}$ is a probability concentrated on $r^{-1}(u) \cap d^{-1}(v)$ for λ'-almost every (u, v) in $E \mid U_0$. The invariance properties imply that this holds for (u, v) in the contraction of E to a (relatively) saturated conull subset of U_0. We may replace U_0 by this subset.

THEOREM 6.9. *Let* $(G, [\lambda])$ *be a measurable groupoid. For almost every unit* u, $G \mid \{u\}$ *has a quasiinvariant measure and hence has a locally compact topology for which it is a topological group and the given measure is equivalent to Haar measure.*

Proof. Choose U_0 according to Theorem 6.8, and suppose that it is λ which has a good decomposition over E. For $u \in U_0$, $\lambda_{u,u}$ is a quasiinvariant probability on $G \mid \{u\}$. Apply Theorem 7.1 of [7].

Remark. It seems natural to ask if one can select a measure λ on G and a decomposition over E such that each $\lambda_{u,u}$ is a Haar measure. This would be in some sense a canonical measure for G.

7. BOOLEAN DUALS OF CONTRACTIONS

Let $(G, [\lambda])$ be a measurable groupoid. In this section we want to characterize in a Boolean fashion the homomorphisms ψ_f discussed in Theorem 6.17 of [14]. If $f: U \to G$ is a function with $d \circ f = i_U$ and $V = r \circ f(U)$, we define $\psi = \psi_f: G \to G \mid V$ by

$$\psi(x) = f(r(x))\, x f(d(x))^{-1}.$$

The goal is to describe the subalgebra of $M(\lambda)$ consisting of elements $q(\psi^{-1}(A))$ where A is a Borel set in $G \mid V$. It will be convenient to begin with some algebraic facts.

THEOREM 7.1. *Let F, G be groupoids and let φ be a homomorphism of G onto F. Then the following conditions are equivalent.*

(a) *There are a $V \subseteq U$ and an $f: U \to G$ with $d \circ f = i_U$, $f(v) = v$ for $v \in V$ and $r \circ f(U) = V$ such that $\varphi \circ \psi_f = \varphi$ and φ is one–one on $G \mid V$.*

(b) *For every $u, v \in U$, φ takes $r^{-1}(u) \cap d^{-1}(v)$ one–one onto $r^{-1}(\tilde{\varphi}(u)) \cap d^{-1}(\tilde{\varphi}(v))$.*

(c) *The function $(\varphi, (r, d))$ takes G one–one onto $F \times' E_G = \{(y, (u, v)): r(y) = \tilde{\varphi}(u)$ and $d(y) = \tilde{\varphi}(v)\}$.*

(d) *For each $y \in F$, (r, d) takes $\varphi^{-1}(y)$ one–one onto $\tilde{\varphi}^{-1}(r(y)) \times \tilde{\varphi}^{-1}(d(y))$.*

Proof. To prove that (a) implies (b) it suffices to prove that (b) holds for $\varphi = \psi_f = \psi$. Let $u_1, u_2 \in U$ and $v_1 = \tilde{\psi}(u_1)$, $v_2 = \tilde{\psi}(u_2)$. Clearly $\psi(r^{-1}(u_1) \cap d^{-1}(u_2)) \subseteq r^{-1}(v_1) \cap d^{-1}(v_2)$. If $d(x) = d(y) = u_2$ and $r(x) = r(y) = u_1$ while $\psi(x) = \psi(y)$, then $f(u_1)\, x f(u_2)^{-1} = f(u_1)\, y f(u_2)^{-1}$ so $x = y$. Thus ψ is one–one on the given set. If $r(y) = v_1$ and $d(y) = v_2$, then $x = f(u_1)^{-1}\, y f(u_2)$ is defined and $\psi(x) = y$, so $\psi(r^{-1}(u_1) \cap d^{-1}(u_2)) = r^{-1}(v_1) \cap d^{-1}(v_2)$.

If (b) holds, let $g: U_F \to U$ be a function for which $\tilde{\varphi} \circ g$ is the identity on U_F, so the set $V = g(U_F)$ meets each level set of $\tilde{\varphi}$ exactly once, i.e., $\tilde{\varphi}$ takes V one–one onto U_F. We show first that φ is one–one on $G \mid V$. If $\varphi(x) = \varphi(y)$ and $x, y \in G \mid V$, then $\tilde{\varphi}(r(x)) = r(\varphi(x)) = r(\varphi(y)) = \tilde{\varphi}(r(y))$ so $r(x) = r(y)$ since both are in V. Similarly $d(x) = d(y)$. Since φ is one–one on $r^{-1}(r(x)) \cap d^{-1}(d(x))$, $x = y$.

For any $u \in U$, φ takes $r^{-1}(g \circ \tilde{\varphi}(u)) \cap d^{-1}(u)$ one–one onto $r^{-1}(\tilde{\varphi}(u)) \cap d^{-1}(\tilde{\varphi}(u))$, so there is a unique $f(u)$ in that former set with $\varphi(f(u)) = \tilde{\varphi}(u)$. Then $d \circ f$ is the identity function of U, $f(v) = v$ for $v \in V$ and $r \circ f(U) = V$, while if $x \in G$, then

$$
\begin{aligned}
\varphi(\psi_f(x)) &= \varphi(f(r(x)))\, \varphi(x)\, \varphi(f(d(x))^{-1}) \\
&= \varphi(r(x))\, \varphi(x)\, \varphi(d(x)) \\
&= \varphi(x).
\end{aligned}
$$

To see that (b), (c), and (d) are equivalent, notice that

$$
\begin{aligned}
&\bigcup \{r^{-1}(\tilde{\varphi}(u)) \cap d^{-1}(\tilde{\varphi}(v)) \times \{(u, v)\}: (u, v) \in E_G\} \\
&\quad = F \times' E_G = \bigcup \{\{y\} \times \tilde{\varphi}^{-1}(r(y)) \times \tilde{\varphi}^{-1}(d(y)): y \in F\}.
\end{aligned}
$$

The function $(\varphi, (r, d))$ is always a homomorphism into $F \times' E_G$ and the one–one onto conditions of (b), (c), and (d) are equivalent because of the two partitions relative to (r, d) and φ given above.

The equivalence of (a) with the other conditions shows that the homomorphisms ψ_f are those for which the partition induced is complementary to the partition induced by (r, d). The rest of this section is devoted to showing that the measure theoretic analog of the equivalence between (a) and (c) is also true.

THEOREM 7.2. *Let (G, C) be a measurable groupoid and λ be a symmetric probability measure in C. Let V be a Borel set of units with $[V] = U$ and let $f\colon U \to G$ be a Borel function such that $d \circ f = i_U$, $f(v) = v$ for $v \in V$ and $r \circ f(U) = V$. Define $\psi_f(x) = f(r(x))\, x f(d(x))^{-1}$ for $x \in G$, $\psi = \psi_f$. If $\mu = \psi_*(\lambda)$ and $\lambda' = (r, d)_*(\lambda)$, then $(\psi, (r, d))$ induces an isomorphism of $M(\mu \times' \lambda')$ onto $M(\lambda)$.*

Proof. If $H = \psi(G) = G \mid V$ and $E = E_G = \{(u, v) \in U \times U\colon u \sim v\}$, we can form the relative product $H \times' E = \{(y, (u, v))\colon r(y) = \psi(u),\ d(y) = \psi(v)\}$ over E_H, and the fibered product $\mu \times' \lambda'$. Now $(\psi, (r, d))$ maps G one–one onto $H \times' E$, so it suffices to prove that $(\psi, (r, d))_*(\lambda) \sim \mu \times' \lambda'$, and by Lemma 1.7 this will follow if $\psi_*(\lambda_{u,v}) \sim \mu_{\psi(u),\psi(v)}$ for λ'-almost every $(u, v) \in E$, where $\lambda = \int \lambda_{u,v}\, d\lambda'(u, v)$ is a decomposition of λ over λ' and $\mu = \int \mu_{a,b}\, d\mu'(a, b)$ is a similar decomposition of μ over μ' on E_H. By passing to an i.c. of G we may assume the decomposition of λ over λ' is quasiinvariant under left and right translations and that $\lambda_{u,v}$ is always concentrated on $r^{-1}(u) \cap d^{-1}(v)$ (Theorem 6.8).

Decompose λ' over μ' relative to $\tilde{\psi} \times \tilde{\psi}$: $\lambda' = \int \lambda'_{a,b}\, d\mu'(a, b)$. Then $\lambda = \int \lambda_{u,v}\, d\lambda'(u, v) = \iint \lambda_{u,v}\, d\lambda'_{a,b}(u, v)\, d\mu'(a, b)$. Now $\lambda'_{a,b}$ is carried by $\tilde{\psi}^{-1}(a) \times \tilde{\psi}^{-1}(b)$ for almost every $(a, b) \in E_H$, and $\lambda_{u,v}$ is carried by $r^{-1}(u) \cap d^{-1}(v)$ for almost every (u, v), while $\psi(r^{-1}(u) \cap d^{-1}(v)) = r^{-1}(a) \cap d^{-1}(b)$ whenever $\psi(u) = a$, $\psi(v) = b$. Thus if we define $\mu_{a,b} = \int \psi_*(\lambda_{u,v})\, d\lambda'_{a,b}(u, v)$ for $(a, b) \in E_H$, then we have $\mu_{a,b}$ carried by $r^{-1}(a) \cap d^{-1}(b)$ for almost every (a, b). Furthermore, $\int \mu_{a,b}\, d\mu'(a, b) = \psi_*(\lambda) = \mu$, so this is a decomposition of μ over μ' relative to (r, d) by Lemma 1.2. Thus the proof will be complete if for μ'-almost every (a, b) we have $\psi_*(\lambda_{u,v}) \sim \mu_{a,b}$ for $\lambda'_{a,b}$-almost every (u, v).

Suppose $\psi(u_1) = \psi(u_2) = a$, $\psi(v_1) = \psi(v_2) = b$, with $a \sim b$. Then there exist x, y in G with $r(x) = u_1$, $d(x) = u_2$, $d(y) = v_1$, $r(y) = v_2$, $\psi(x) = a$, $\psi(y) = b$. From quasiinvariance of the $\lambda_{u,v}$, it follows that $\psi_*(\lambda_{u_1,v_1}) \sim \psi_*(\lambda_{u_2,v_2})$. Thus if $\lambda'_{a,b}$ is carried by $\tilde{\psi}^{-1}(a) \times \tilde{\psi}^{-1}(b)$, $\psi_*(\lambda_{u,v}) \sim \mu_{a,b}$ for $\lambda'_{a,b}$-almost every (u, v) (namely all in $\tilde{\psi}^{-1}(a) \times \tilde{\psi}^{-1}(b)$). Now $\lambda'_{a,b}$ is carried by $\tilde{\psi}^{-1}(a) \times \tilde{\psi}^{-1}(b)$ for μ'-almost every (a, b), so we have the desired conclusion.

THEOREM 7.3. *Let $(G, [\lambda])$ and $(F, [\mu])$ be measurable groupoids and let $\varphi: G \to F$ be a homomorphism with $\varphi_*(\lambda) \sim \mu$. Suppose $(\varphi, (r, d))$: $G \to F \times' E_G$ induces an isomorphism of $M(\mu \times' \lambda')$ onto $M(\lambda)$. Then there are a Borel set $V \subseteq U$ and a Borel function $f: U \to G$ such that $d \circ f$ is the identity on U, $f(v) = v$ for $v \in V$, $V = r \circ f(U)$, and φ establishes an isomorphism between i.c.'s of $(G \mid V, [\psi_{f*}(\lambda)])$ and $(F, [\mu])$, while $\varphi \circ \psi_f = \varphi$ on an i.c. of G.*

Proof. (The method is to reduce to the algebraic case.) We take a decomposition of λ over λ' relative to (r, d) as in Theorem 6.8 and do the same for μ over μ', supposing the integrands quasiinvariant and concentrated on the proper sets. Then $E_1 = \{(u, v) \in E: \varphi_*(\lambda_{u,v}) \sim \mu_{\varphi(u),\varphi(v)}\}$ is a Borel set, being the inverse image of a Borel set (Lemma 1.1) under the Borel mapping $(u, v) \to (\varphi_*(\lambda_{u,v}), \mu_{\varphi(u),\varphi(v)})$. Lemma 1.7 shows that E_1 is conull, since $(\varphi, (r, d))_*(\lambda) \sim \mu \times' \lambda'$ is assumed here. If $(u, v) \in E_1$ and $(v, w) = (r, d)(x) \in E_1$, then

$$\begin{aligned}\varphi_*(\lambda_{u,w}) &\sim \varphi_*(\lambda_{u,v} \cdot x)\\ &= \varphi_*(\lambda_{u,v}) \cdot \varphi(x)\\ &\sim \mu_{\varphi(u),\varphi(v)} \cdot \varphi(x)\\ &\sim \mu_{\varphi(u),\varphi(w)} .\end{aligned}$$

Hence $(u, w) \in E_1$, so E_1 is closed under multiplication and must contain an i.c. (In fact we can contract to a conull saturated set, but this is not necessary for the proof.) Thus by passing to an i.c. we can continue the proof under the assumption that $\varphi_*(\lambda_{u,v}) \sim \mu_{\varphi(u),\varphi(v)}$ for all $(u, v) \in E$.

For $u \in U_G$ or $w \in U_F$ let $G_u = r^{-1}(u) \cap d^{-1}(u)$, and $F_w = r^{-1}(w) \cap d^{-1}(w)$. By Theorem 6.9, G_u and F_w "are" locally compact groups with $\lambda_{u,u}$ and $\mu_{w,w}$ equivalent to Haar measure. For any $u \in U$, $\varphi_*(\lambda_{u,u}) \sim \mu_{\varphi(u),\varphi(u)}$ so $\varphi(G_u)$ is a conull analytic (and hence measurable) subgroup of $F_{\varphi(u)}$ and hence must equal $F_{\varphi(u)}$. If $r(x) = v$, $d(x) = u$ and $z \in r^{-1}(\varphi(u)) \cap d^{-1}(\varphi(v))$, then $r(\varphi(x)) = d(z)$ and $z\varphi(x) \in F_{\varphi(u)}$. Thus there is a $y \in G_u$ with $\varphi(y) = z\varphi(x)$ and hence $\varphi(yx^{-1}) = z$. Thus φ takes $r^{-1}(u) \cap d^{-1}(v)$ onto $r^{-1}(\varphi(u)) \cap d^{-1}(\varphi(v))$.

By Lemma 1.7, there must be a conull Borel set $K \subseteq G$ on which $(\varphi, (r, d))$ is one–one. Then K is $\lambda_{u,v}$-conull for almost every (u, v), so φ is one–one a.e. on almost every $r^{-1}(u) \cap d^{-1}(v)$. If φ is one–one a.e. on $r^{-1}(u) \cap d^{-1}(v)$ $(v \in [u])$ and $r(x) = v$, then φ is one–one a.e. on $r^{-1}(u) \cap d^{-1}(d(x))$ because $\lambda_{u,v} \cdot x \sim \lambda_{u,d(x)}$ and $y \to yx^{-1} \to \varphi(yx^{-1}) = \varphi(y)\,\varphi(x)^{-1} \to \varphi(y)$ is a composition of one–one functions. Thus for any u, $\{v \in [u]\colon \varphi$ is one–one a.e. on $r^{-1}(u) \cap d^{-1}(v)\}$ is either $\varnothing$ or $[u]$. Now for almost every u, K is $\lambda_{u,v}$-conull for $r_*(\lambda_u)$-almost every v, so there is a conull (saturated) set of units U_0 such that $(u, v) \in E \mid U_0$ implies φ is one–one a.e. on $r^{-1}(u) \cap d^{-1}(v)$. Again contract to an i.c., so that φ is one–one a.e. on every $r^{-1}(u) \cap d^{-1}(v)$. If $A \subseteq G_u$ is a conull set on which φ is one–one and $x, y \in G_u$, then $Ax^{-1} \cap Ay^{-1} \neq \varnothing$ because both sets are conull. If z is in the intersection then zx and zy are in A. Now $\varphi(x) = \varphi(y)$ implies $\varphi(zx) = \varphi(zy)$ so $zx = zy$ and hence $x = y$. Thus φ is one–one on G_u. It follows that φ is actually one–one on every $r^{-1}(u) \cap d^{-1}(v)$.

At this point, $\varphi(G)$ is a conull subset of F, closed under multiplication, and hence contains an i.c. F_0. Now $\varphi^{-1}(F_0)$ is also an i.c. of G, so by taking one more i.c. we can put ourselves in the setting of a Borel homomorphism of G onto F in which φ takes $r^{-1}(u) \cap d^{-1}(v)$ one–one onto $r^{-1}(\varphi(u)) \cap d^{-1}(\varphi(v))$ for every $(u, v) \in E$.

Let $g\colon U_F \to U_G$ be a Borel function such that $\tilde{\varphi} \circ g$ agrees a.e. with the identity on U_F. By passing to yet another i.c. (first F, then G, as above) we may suppose that $\tilde{\varphi} \circ g$ is the identity on U_F. Now F is countably separated so the set where r and $g \circ \tilde{\varphi} \circ d$ agree is a Borel set and so is $\varphi^{-1}(U_F)$. Let their intersection be denoted by B. Now if $\varphi(x)$ is a unit, that unit is $\varphi(d(x))$. Let $x, y \in B$ with $d(x) = d(y)$. Then $r(x) = g \circ \tilde{\varphi} \circ d(x) = r(y)$, and since $\varphi(x)$ and $\varphi(y)$ are units, $\varphi(x) = \varphi(y)$. Since φ is one–one on $r^{-1}(r(x)) \cap d^{-1}(d(x))$, $x = y$. Thus d takes B one–one onto U and since d is Borel the inverse, $f = (d \mid B)^{-1}$, is Borel. Now $f\colon U \to G$ is the same function as appeared in the proof of Theorem 7.1, so the algebraic part of that proof applies. Thus φ is an isomorphism of $G \mid g(U_F)$ onto F and $\varphi \circ \psi_f = \varphi$. The last equation implies that $\varphi_*(\psi_{f*}(\lambda)) = \varphi_*(\lambda) = \mu$, so the isomorphism preserves measure classes also.

To get back to all of G as in the statement of the theorem, extend f to be the identity on the rest of the original U, and then the set $g(U_F)$ just above is conull in $r \circ f(U)$, which we can take for V.

THEOREM 7.4. *Let* $B = (M, \tilde{M}, \delta, (\ )^{-1}, \mu)$ *be a Boolean cogroupoid and* $B_1 = (M_1, \tilde{M}_1, \delta_1, (\ )^{-1}, \mu_1)$ *the Boolean cogroupoid dual to a quotient groupoid, with* $M_1 \subseteq M$ *and the inclusion dual to the quotient homomorphism. Let* M_e *denote the subalgebra of* M *corresponding to the quotient equivalence relation. Then there is a relative tensor product* $M_1 \otimes' M_e$, *and* B_1 *corresponds to a contraction of* G

iff the inclusion homomorphisms of M_1, M_e into M extend to an isomorphism of $M_1 \otimes' M_e$ onto M.

Proof. It is easy to see that $M_{1e} = \delta_1(\tilde{M}_1) \vee \delta_1(\tilde{M}_1)^{-1}$ is contained in $M_e = \delta(\tilde{M}) \vee \delta(\tilde{M})^{-1}$, so the relative tensor product $M_1 \otimes' M_e$ over M_{1e} can be formed. The rest of the statement is simply the dual of Theorems 7.2 and 7.3.

8. Boolean Duals of Homomorphisms (III)

The complete description of homomorphisms can now be given by combining the results in Sections 5 and 7. The range of a homomorphism φ must contain a contraction which is the range of a ψ_f as described in Section 7. That is, we can reduce to the case in which $\tilde{\varphi}^*$ exists and is one–one. In other words, to get a homomorphism from F to G one first selects a subalgebra M_1 of $M_G = M(\mu)$ of the type corresponding to a contraction, $M_1 = M_{G|V} = M(\mu_1)$. Next one chooses an imbedding γ: $M(\tilde{\mu}_1) \to M(\tilde{\lambda})$ where $M(\lambda) = M_F$, and a σ-homomorphism α: $M(\mu_1) \to M(\lambda) \otimes' M(\mu_1)$ so that the pair (γ, α) satisfies (5.1b) to (5.4b). Theorem 5.4 says that every homomorphism $\varphi: F \to G \mid V$ for which $\tilde{\varphi}^*$ exists and is one–one arises from such a pair, essentially once.

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